Crossed products and cleft extensions for coquasi-Hopf algebras

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Summary

1. *Coquasi-bialgebras, crossed products and various associated categories*
2. *Cleft extensions and crossed products*
    - *Computation of some crossed products in low dimension*

Coquasi-bialgebras and coquasi-Hopf algebras

- (Majid (1992), Panaite, Ştefan(1997)) H **coquasi-bialgebra**
  - coassociative coalgebra Δ, ϵ
  - $\exists$ unit and multiplication, no longer associative
  - associativity of multiplication controlled by $\omega \in (H \otimes H \otimes H)^*$

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- **Monoidal category** (but not strict!) of comodules $(\mathcal{M}^H, \otimes)$
- H **coquasi-Hopf algebra** if it exists also an antimorphism of coalgebras (antipode) S and elements $\alpha, \beta \in H^*$ with some properties which ensure the rigidity of $\mathcal{M}^H_{f\,\dim}$

Data for crossed products

H coquasi-bialgebra, R associative algebra

- Weak action $\circ : H \otimes R \longrightarrow R$:

$$h \circ (rs) = (h_1 \circ r)(h_2 \circ s), \quad h \circ 1_R = \varepsilon(h) 1_R$$

- Convolution invertible map $\sigma : H \otimes H \longrightarrow R$.

Definition

The **crossed product** $R \#_\sigma H$ *is* $R \otimes H$ *with multiplication*

$$(r \#_\sigma h)(s \#_\sigma g) = r(h_1 \circ s)\sigma(h_2, g_1) \#_\sigma h_3 g_2$$

Crossed products as comodule algebras

Theorem

$R\#_\sigma H$ is an H-comodule algebra (i.e. algebra in monoidal category $\mathcal{M}^H$) with unit $1_R\#_\sigma 1_H$ and coaction $I_R\otimes\Delta$ if and only if

$$\begin{aligned}
1_H\circ r &= r\\
[h_1\circ(g_1\circ r)]\sigma(h_2,g_2) &= \sigma(h_1,g_1)[(h_2g_2)\circ r]\\
\sigma(h,1) &= \sigma(1,h)=\varepsilon(h)1_R\\
[h_1\circ\sigma(g_1,l_1)]\sigma(h_2,g_2l_2) &= \sigma(h_1,g_1)\sigma(h_2g_2,l_1)\omega^{-1}(h_3,g_3,l_2)
\end{aligned}$$

In this case we say that $(R,\circ,\sigma)$ form a **crossed H-system**.

Crossed products

- Any crossed product $R\#_\sigma H$ by a bialgebra H
- Trivial cocycle implies ω trivial (H is a bialgebra) and R is a left H-module algebra $\Longrightarrow$ usual smash product $R\#H$
- Trivial weak action implies $\mathrm{Im}\,\sigma \subseteq Z(R)$ and $\omega = \partial\sigma \Longrightarrow$ twisted product $R_\sigma[H]$ with multiplication $(r\#_\sigma h)(s\#_\sigma g) = rs\sigma(h_1, g_1)\#_\sigma h_2 g_2$
- $R = \Bbbk$: weak action must be trivial $\Longrightarrow$ H is a deformation of a bialgebra by the twist σ

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- H coquasi-bialgebra, $(H^*, *, \rightharpoonup)$ associative algebra

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  - $Hom(H,A)$ associative algebra

$$(\lambda \circledast \kappa)(h) = \lambda(\kappa(h_3)_2 h_2)_0 \kappa(h_3)_0 \omega^{-1}(\lambda(\kappa(h_3)_2 h_2)_1, \kappa(h_3)_1, h_1)$$

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  - Weak action

$$(h \circ \lambda)(g) = \lambda(gh)$$

Cocycle

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Crossed system $(Hom(H, A), \circ, \sigma)$

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    Crossed system $(Hom(H, A), \circ, \sigma)$
  - Possible duality theorem $(A \# H^*) \# H \simeq M_{\dim_{\Bbbk} H}(A)$

Examples of crossed products

- $R = \mathbb{k}$ with trivial action, H_σ deformation of a bialgebra $H \Longrightarrow$ crossed product $\mathbb{k}\#_{\sigma^{-1}}(H_\sigma)$
- $H = \mathbb{k}G$, for G finite group $\Longrightarrow$ quasialgebras $\mathbb{k}\#_{\sigma^{-1}}(\mathbb{k}G_\sigma)$
- $G = (\mathbb{Z}_2)^n \Longrightarrow$ all Cayley and Clifford algebras can be obtained as $\mathbb{k}\#_{\sigma^{-1}}(\mathbb{k}(\mathbb{Z}_2)^n)_\sigma$

Crossed products viewed towards monoidal categories

H coquasi-bialgebra, R associative algebra

Proposition

TFAE:

1. *There exist a weak action of H on R and a convolution invertible map $\sigma : H \otimes H \longrightarrow R$ such that $(R, \circ, \sigma)$ is crossed H-system*

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5. $({}^H\mathcal{M}_R, \otimes)$ *is right* ${}^H\mathcal{M}^H$*-category*

- From a crossed system $(R, \circ, \sigma)$

$$\begin{array}{rcl} (M,V) & \in & \mathcal{M}_R \times {}^H\mathcal{M} \longrightarrow M \otimes V \in \mathcal{M}_R \\ (m \otimes v)r & = & m(v_{-1} \circ r) \otimes v_0 \\ \Psi_{M,V,W} & : & (M \otimes V) \otimes W \longrightarrow M \otimes (V \otimes W) \\ (m \otimes v) \otimes w & \longrightarrow & m\sigma(v_{-1}, w_{-1}) \otimes (v_0 \otimes w_0) \end{array}$$

- From a right ${}^H\mathcal{M}$-category action $(\mathcal{M}_R, \otimes, \Psi)$

$$\begin{array}{rcl} h \circ r & = & (I_R \otimes \varepsilon)((1_R \otimes h)r) \\ \sigma(h,g) & = & (I_R \otimes \varepsilon \otimes \varepsilon)\Psi_{R,H,H}(1_R \otimes h \otimes g) \end{array}$$

Twist invariance of the crossed product
(changing the monoidal category)

- H coquasi-bialgebra
- τ twist on H
- $(R, \circ, \sigma)$ crossed H-system

$\Longrightarrow$

- Deformed coquasi-bialgebra H_τ
- Deformed cocycle $\sigma\tau^{-1}$
- $(R, \circ, \sigma\tau^{-1})$ crossed H_τ-system

Proposition

$$R\#_{\sigma\tau^{-1}} H_\tau = (R\#_\sigma H)_{\tau^{-1}}$$

Twist transformation of the crossed product

(changing the action of the monoidal category)

- H coquasi-bialgebra
- $(R, \circ, \sigma)$ crossed H-system
- $\upsilon : H \longrightarrow R$ convolution invertible map

$\Longrightarrow$

- New weak action $\circ_\upsilon$ and new convolution invertible map $\sigma_\upsilon : H \otimes H \longrightarrow R$
- $(R, \circ_\upsilon, \sigma_\upsilon)$ crossed H-system
- $R\#_{\sigma_\upsilon}H$ **twist transformation** of $R\#_\sigma H$

When two crossed products are isomorphic?

Proposition

- *H coquasi-bialgebra*
- *R associative algebra*
- *Two crossed systems $(R, \circ_1, \sigma_1)$ and $(R, \circ_2, \sigma_2)$*

$R\#_{\sigma_1}H \simeq R\#_{\sigma_2}H$ isomorphism of right H-comodule algebras, left R-modules $\Longleftrightarrow$ $R\#_{\sigma_2}H$ is a twist transformation of $R\#_{\sigma_1}H$

Examples of crossed products

- $\mathfrak{H}$ f. d. quasi-bialgebra $(\mathcal{A}, \rho, \phi_\rho)$ $\mathfrak{H}$-comodule algebra
  $\Downarrow$

$$\Longleftrightarrow$$

Examples of crossed products

- $\mathfrak{H}$ f. d. quasi-bialgebra
  $(\mathcal{A}, \rho, \phi_\rho)$ $\mathfrak{H}$-comodule algebra
  $\Downarrow$
- Cocycle
  $\sigma(\mathfrak{h}^*, \mathfrak{g}^*) = x_\rho^1 \mathfrak{h}^*(x_\rho^2)\mathfrak{g}^*(x_\rho^3) \iff$
  Weak action
  $\mathfrak{h}^* \circ r = (I \otimes \mathfrak{h}^*)\rho(r)$
  Crossed $\mathfrak{H}^*$-system $(\mathcal{A}, \circ, \sigma)$
  Crossed product
  $\mathcal{A}\#_\sigma \mathfrak{H}^* = \mathcal{A}\overline{\#}\mathfrak{H}^*$

Examples of crossed products

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  Weak action $\mathfrak{h}^* \circ r = (I \otimes \mathfrak{h}^*)\rho(r)$
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  $\sigma(\mathfrak{h}^*, \mathfrak{g}^*) = x^1_\rho \mathfrak{h}^*(x^2_\rho)\mathfrak{g}^*(x^3_\rho)$ $\Longleftrightarrow$
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- H f. d. coquasi-bialgebra $(R, \circ, \sigma)$ assoc. algebra
  $\Downarrow$
- Coaction
  $$\rho(r) = \sum_{i=1}^{\dim_{\Bbbk} H} e_i \circ r \otimes e^i$$
  Associator
  $$\phi_\rho = \sum_{i,j=1}^{\dim_{\Bbbk} H} \sigma^{-1}(e_i, e_j) \otimes e^i \otimes e^j$$
  H^*-comodule algebra (R, ρ, ϕ_ρ)

Coquasi-Hopf modules

Definition

Category of **right coquasi-Hopf modules** $(\mathcal{M}_R^H)_H$ is the category of right H-modules in $\mathcal{M}_R^H$
Category of **left coquasi-Hopf modules** ${}_R^H\mathcal{M}_H$ is the category of right H-modules in ${}_R^H\mathcal{M}$

Coquasi-Hopf modules

Theorem

H coquasi-bialgebra, $(R, \circ, \sigma)$ crossed H-system. Then the category of coquasi-Hopf modules is isomorphic to the category of relative $(H, R\#_\sigma H)$-Hopf modules

$$\begin{aligned} (\mathcal{M}_R^H)_H &\simeq \mathcal{M}_{R\#_\sigma H}^H \\ {}_R^H\mathcal{M}_H &\simeq {}_{R\#_\sigma H}\mathcal{M}^H \end{aligned}$$

Second isomorphism much more complicated and uses antipode (bijective!) and Drinfeld's twist.

Coquasi-Hopf modules

- (Bulacu, Caenepeel) $\mathfrak{H}$ finite dimensional quasi-Hopf algebra (**with antipode!**)
  $\mathcal{A}$ right $\mathfrak{H}$-comodule algebra
  Category of two-sided Hopf modules ${}_{\mathfrak{H}}\mathcal{M}_{\mathcal{A}}^{\mathfrak{H}}$
- Isomorphism of categories ${}_{\mathfrak{H}}\mathcal{M}_{\mathcal{A}}^{\mathfrak{H}} \simeq {}_{\mathfrak{H}}\mathcal{M}_{\mathcal{A}\overline{\#}\mathfrak{H}^*}$
- For $H = \mathfrak{H}^*$ and $R = \mathcal{A}$
  Isomorphism between the two-sided Hopf modules and coquasi-Hopf modules

$$ {}_{\mathfrak{H}}\mathcal{M}_{\mathcal{A}}^{\mathfrak{H}} \simeq (\mathcal{M}_{\mathcal{A}}^{\mathfrak{H}^*})_{\mathfrak{H}^*} $$

  Not explained by an algebra-coalgebra duality between $\mathfrak{H}$ and $H = \mathfrak{H}^*$ in monoidal category ${}_{\mathfrak{H}}\mathcal{M}_{\mathfrak{H}}$
- Our formulas more natural, not requiring finite dimension or existence of the antipode

Coquasi-Hopf modules

- Now H coquasi-Hopf algebra (with bijective antipode in left case)
$M \in {}^H_R\mathcal{M}_H \qquad M^{coH} = \{m \in M \,|\, \lambda(m) = 1_H \otimes m\}$
$M \in (\mathcal{M}^H_R)_H \qquad M^{coH} = \{m \in M \,|\, \rho(m) = m \otimes 1_H\}$
- Projection on coinvariants

$$\begin{array}{lll} M & \in & {}^H_R\mathcal{M}_H \qquad \Pi_l(m) = m_0 \circledcirc [\alpha \rightharpoonup S^{-1}(m_{-1})] \\ M & \in & (\mathcal{M}^H_R)_H \qquad \Pi_r(m) = m_0 \circledcirc S(m_1 \leftharpoonup \beta) \end{array}$$

Coquasi-Hopf modules

Theorem

For any coquasi-Hopf algebra H (with bijective antipode in the left case) and any crossed system $(R, \circ, \sigma)$, the category of coquasi-Hopf modules is equivalent to the category of R-modules

$$\mathcal{M}_R \underset{(-)^{coH}}{\overset{-\otimes H}{\rightleftarrows}} (\mathcal{M}_R^H)_H \qquad {}_R\mathcal{M} \underset{(-)^{coH}}{\overset{-\otimes H}{\rightleftarrows}} {}_R^H\mathcal{M}_H$$

Doi-Hopf modules...

- H coquasi-bialgebra, $(R, \circ, \sigma)$ crossed system $\Longleftrightarrow$ $\mathcal{M}_R$ is right ${}^H\mathcal{M}$-category with tensor product over the base field
- C left H-comodule coalgebra $\Longrightarrow$ right C-comodules in $\mathcal{M}_R$
- Category of Doi-Hopf modules $\mathcal{M}_R^C$

...and a coring associated to a crossed system

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Proposition

$\mathcal{C} = {}_\bullet R_\bullet \otimes C_\bullet$ is an R-coring and the category of Doi-Hopf modules $\mathcal{M}_R^C$ is isomorphic to the category $\mathcal{M}^{\mathcal{C}}$ of right comodules over $\mathcal{C}$

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Particular case $R = H^*$ and $C = \overline{H}$.
Then what is the category of Doi-Hopf modules $\mathcal{M}_{H^*}^{\overline{H}}$?

H coquasi-bialgebra

- A right H-**comodule algebra** A is an algebra in the monoidal category $\mathcal{M}^H$.
- A is no longer associative!
- Algebra (associative) of coinvariants $B = A^{coH}$

Cleft extensions

Definition

A right H-comodule algebra is **cleft** *if* $\exists\ \gamma, \delta : H \longrightarrow A$ such that

$$\rho(\gamma(h)) = \gamma(h_1) \otimes h_1 \qquad \rho(\delta(h)) = \delta(h_2) \otimes S(h_1)$$
$$\delta(h_1)\gamma(h_2) = \alpha(h)1_A \qquad \gamma(h_1)\beta(h_2)\delta(h_3) = \varepsilon(h)1_A$$

Cleft extensions

Proposition

There is a Morita context $\mathbb{M}(A)$ *with*

- *rings* $Hom^H(\overline{H}, A)$ *and* $Hom(H, B)$
- *bimodules* $Hom^H(H, A)$ *and* $Hom^H(H^S, A)$
- *connecting morphisms*

$$\begin{aligned}
(-,-) &: Hom^H(H, A) \otimes_{Hom^H(\overline{H},A)} Hom^H(H^S, A) \longrightarrow Hom(H, B) \\
(\mathfrak{p}, \mathfrak{q})(h) &= \mathfrak{p}(h_1)\beta(h_2)\mathfrak{q}(h_3) \\
[-,-] &: Hom^H(H^S, A) \otimes_{Hom(H,B)} Hom^H(H, A) \longrightarrow Hom^H(\overline{H}, A) \\
[\mathfrak{q}, \mathfrak{p}](h) &= \mathfrak{q}(h_1)\mathfrak{p}(h_2)
\end{aligned}$$

Cleft extensions

Theorem

H coquasi-Hopf algebra, A right H-comodule algebra $\Longrightarrow$

1. *First Morita map* $[,]$ *is surjective* $\Longleftrightarrow$
   $$\begin{cases} B \subseteq A \text{ is Galois} \\ \exists\ n > 0, \text{ s. t. } A \text{ is direct summand in } ({}_\bullet B \otimes H^\bullet)^n \end{cases}$$
2. *Strict Morita context* $\Longleftrightarrow$
   $$\begin{cases} \bullet\ B \subseteq A \text{ is Galois} \\ \bullet\ \exists\ n > 0, \text{ s. t. } A \text{ is direct summand in } ({}_\bullet B \otimes H^\bullet)^n \\ \bullet\ \exists\ n, r > 0 \text{ s. t. } A \text{ is direct summand in } ({}_\bullet B \otimes H^\bullet)^n \text{ and} \\ \qquad {}_\bullet B \otimes H^\bullet \text{ is direct summand in } A^r \end{cases}$$
3. $B \subseteq A$ *cleft extension* $\Longrightarrow$ *strict Morita context.*

Crossed products are the same as cleft extensions

Theorem

H coquasi-Hopf algebra, A right H-comodule algebra. TFAE:

1. $B \subseteq A$ *is cleft*
2. $A \simeq B\#_\sigma H$ *as right H-comodule algebras, left B-modules*

Cleaving maps

$$\gamma(h) = 1_A \#_\sigma h \qquad \delta(h) = \sigma^{-1}(S(h_2), h_3 \leftharpoonup \alpha)\#_\sigma S(h_1)$$

Weak action and cocycle

$$\begin{aligned} h \circ b &= \gamma(h_1) b \delta(h_2 \leftharpoonup \beta) \\ \sigma(h, g) &= [\gamma(h_1)\gamma(g_1)]\delta((h_2 g_2) \leftharpoonup \beta) \end{aligned}$$

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- For coquasi-Hopf algebras, does not work
- If yes, it would mean that $A = \mathbb{k}\#_\sigma H$, impossible unless H is a deformation of a Hopf algebra
- However...
- $\exists$ similarity between the formulas of the antipode and the cleft extension relations $\Longrightarrow$ $A = H$ cleft over $\mathbb{k}$, in particular Galois, in an appropriate context

Crossed products over coquasi-Hopf algebras of low dimension

- $H(2)$ is the unique non-trivial coquasi-Hopf algebra of dimension 2 over a field $\mathbb{k}$, $char\mathbb{k} \neq 2$, s. t.
  - It has the Hopf algebra structure of $\mathbb{k}[\mathbb{Z}_2]$, $\mathbb{Z}_2 = \{1, x\}$
  - The cocycle $\omega(x, x, x) = -1$
  - The elements α, $\beta \in H(2)^*$ given by $\alpha(1) = 1$, $\alpha(x) = -1$ and β trivial

Crossed products over coquasi-Hopf algebras of low dimension

Proposition

Let R associative algebra. Then $\exists$ $(R, \circ, \sigma)$ crossed $H(2)$-system $\Longleftrightarrow$ $\exists$ $\mathcal{F} \in Aut_{\Bbbk-a\lg}(R)$, $c \in U(R)$ such that

1. $\mathcal{F}^2(e) = cec^{-1}$
2. $\mathcal{F}(c) = -c$

Denote $\left(\dfrac{\mathcal{F}, c}{R}\right)$ the data associated to the crossed product.

Crossed products over coquasi-Hopf algebras of low dimension

Proposition

$\left(\frac{\mathcal{F}, c}{R}\right) \simeq \left(\frac{\mathcal{F}', c'}{R}\right)$ *as right* $H(2)$*-comodule algebras and left* R*-modules* $\Longleftrightarrow \exists\ s \in U(R)$ *s.t.*

$$c' = s^{-1}\mathcal{F}(s)^{-1}c \qquad \mathcal{F}'(e) = s^{-1}\mathcal{F}(e)s$$

Crossed products over coquasi-Hopf algebras of low dimension

- R commutative. Then all crossed systems are completely described by $\mathcal{F} \in Aut_{\Bbbk-a\lg}(R)$ and $c \in U(R)$ such that $\mathcal{F}^2 = I_R$ and $\mathcal{F}(c) = -c$
- R central simple $\Bbbk$-algebra. There are no $H(2)$-crossed products of R.

Crossed products over coquasi-Hopf algebras of low dimension

- $H(3)$ not trivial coquasi-Hopf algebra of dimension 3, basis $\{1, x, x^2\}$, built on $\mathbb{k}[\mathbb{Z}_3]$ with
  - Cocycle ω (Albuquerque, Majid, 1999)

$$\begin{aligned} \omega(x,x,x) &= \omega(x,x,x^2) = 1 \\ \omega(x,x^2,x) &= \omega(x^2,x,x) = \omega(x^2,x^2,x) = q^{-1} \\ \omega(x,x^2,x^2) &= \omega(x^2,x,x^2) = \omega(x^2,x^2,x^2) = q \end{aligned}$$

  for $q \neq 1$ cubic root of 1
  - Elements $\alpha, \beta \in H(3)^*$, α trivial and β given by $\beta(1) = 1, \beta(x) = q, \beta(x^2) = q^{-1}$

Crossed products over coquasi-Hopf algebras of low dimension

Proposition

Let R associative algebra. Then $\exists$ $(R, \circ, \sigma)$ crossed $H(3)$-system $\Longleftrightarrow$ $\exists$ $\mathcal{F}, \mathcal{G} \in \mathrm{Alg}_{\mathbb{k}}(R)$ and $u^{(1)}$, $u^{(2)}$, $v^{(1)}$, $v^{(2)} \in U(R)$ s. t.

$\circ$	$\mathcal{F}$	$\mathcal{G}$
$\mathcal{F}$	$u^{(1)}\mathcal{G}(-)u^{(1)-1}$	$v^{(1)}(-)v^{(1)-1}$
$\mathcal{G}$	$v^{(2)}(-)v^{(2)-1}$	$u^{(2)}\mathcal{F}(-)u^{(2)-1}$

	$u^{(1)}$	$u^{(2)}$	$v^{(1)}$	$v^{(2)}$
$\mathcal{F}$	$u^{(1)}v^{(2)}v^{(1)-1}$	$v^{(1)}u^{(1)-1}q^{-1}$	$u^{(1)}u^{(2)}$	$v^{(1)}q$
$\mathcal{G}$	$v^{(2)}u^{(2)-1}q$	$u^{(2)}v^{(1)}v^{(2)-1}q^{-1}$	$v^{(2)}q^{-1}$	$u^{(2)}u^{(1)}q$

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- Denote by $\left(\dfrac{\mathcal{F}, \mathcal{G}, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}}{R}\right)$ the data needed for a crossed system $(R, \circ, \sigma)$
- $\mathcal{F}\mathcal{G}$, $\mathcal{G}\mathcal{F}$, $\mathcal{F}^3$ and $\mathcal{G}^3$ are inner
- R commutative implies $\mathcal{F}^3 = I_R$ and $\mathcal{F}^2 = \mathcal{G}$.

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Proposition

$$\left(\frac{\mathcal{F}, \mathcal{G}, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}}{R}\right) \simeq \left(\frac{\mathcal{F}', \mathcal{G}', u'^{(1)}, u'^{(2)}, v'^{(1)}, v'^{(2)}}{R}\right) \text{ as}$$

$H(3)$-comodule algebras and left R-modules $\Longleftrightarrow \exists\, s^{(1)}, s^{(2)} \in U(R)$ *s.t.*

$$\begin{aligned}
\mathcal{F}'(e) &= s^{(1)-1}\mathcal{F}(e)s^{(1)} & \mathcal{G}'(e) &= s^{(2)-1}\mathcal{G}(e)s^{(2)} \\
u'^{(1)} &= s^{(1)-1}\mathcal{F}(s^{(1)-1})u^{(1)}s^{(2)} & v'^{(1)} &= s^{(1)-1}\mathcal{F}(s^{(2)-1})v^{(1)} \\
u'^{(2)} &= s^{(2)-1}\mathcal{G}(s^{(2)-1})u^{(2)}s^{(1)} & v'^{(2)} &= s^{(2)-1}\mathcal{G}(s^{(2)-1})v^{(2)}
\end{aligned}$$