Galois extensions for coquasi-Hopf algebras

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Summary

1. Coquasi-Hopf algebras and their coactions
2. Galois extensions and structure theorems for relative Hopf modules
    1. Cleft extensions
3. Galois extensions and a Hopf algebroid construction

Coquasi-Hopf algebras and their coactions

- (Majid (1992), Panaite, Ştefan(1997)) H **coquasi-Hopf algebra**
  - coassociative coalgebra Δ, ϵ
  - $\exists$ unit and multiplication, no longer associative
  - associativity of multiplication controlled by $\omega \in (H \otimes H \otimes H)^*$
  - $\exists$ antimorphism of coalgebras (antipode) S, elements α, $\beta \in H^*$

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- **Monoidal category** (but not strict!) of comodules $(\mathcal{M}^H, \otimes)$

Coquasi-Hopf algebras and their coactions

H coquasi-Hopf algebra.

- A right H-**comodule algebra** A is an algebra in the monoidal category $\mathcal{M}^H$.
- A is no longer associative!
- The space of coinvariants $B = A^{coH} = \{a \in A \,|\, \rho_A(a) = a \otimes 1_H\}$ is an algebra (associative!)

Coquasi-Hopf algebras and their coactions

- (Bulacu, Nauwelaerts, 2000) Right relative (H, A)-**Hopf module** is a right A-module in $\mathcal{M}^H$.
- The category of relatives Hopf modules $\mathcal{M}_A^H$
- Adjunction of categories: $\mathcal{M}_B \underset{(-)^{coH}}{\overset{(-)\otimes_B A}{\rightleftarrows}} \mathcal{M}_A^H$ with counit ε_M

Coquasi-Hopf algebras and their coactions

- **Twist invariance**: take τ twist on H.
  Then H_τ is a coquasi-Hopf algebra, with new multiplication $g \cdot_\tau h = \tau(g_1, h_1) g_2 h_2 \tau^{-1}(g_3, h_3)$, same unit, new cocycle
- $A_{\tau^{-1}}$ is a comodule algebra over H_τ, but with new operation $g \circ_\tau h = g_1 h_1 \tau^{-1}(g_2, h_2)$
- Category isomorphism $\mathcal{M}_A^H \simeq \mathcal{M}_{A_{\tau^{-1}}}^{H_\tau}$

Coquasi-Hopf algebras and their coactions

- Easy way to produce comodule algebras:
  Start with H Hopf algebra. Then $A = H$ is a comodule algebra over itself. Take now τ twist on H. Then $A_{\tau^{-1}}$ is comodule algebra over coquasi-Hopf algebra H_τ

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Coquasi-Hopf algebras and their coactions

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- Apply to the Hopf algebra $H = \Bbbk G$, for G finite group, and twist induced by a 2-cocycle $\sigma : G \times G \longrightarrow \mathbf{k}^*$
- For $G = (\mathbb{Z}_2)^n$: all Cayley and Clifford algebras as comodule algebras over some coquasi-Hopf algebra (Albuquerque, Majid1998-2000)

Galois extensions and structure theorems for relative Hopf modules

H coquasi-Hopf algebra, A right comodule algebra, $B = A^{coH}$

Definition (Balan, 2007)

Galois extension $B \subseteq A \iff$ *bijective map*

$$\begin{aligned} can \quad &: \quad A \otimes_B A \longrightarrow A \otimes H \\ a \otimes_B b \quad &\longrightarrow \quad a_0 b_0 \otimes \omega^{-1}(a_1, b_1 \beta(b_2), S(b_3)) b_4 \end{aligned}$$

Example (Masuoka, 2003)

From operator algebras theory

Matched pair of finite groups $F, G \Longrightarrow$ construction of a coquasi-Hopf algebra $H = \widehat{G}\#_{\sigma,\tau}\mathbb{C}F$ as a bicrossproduct with some cocycle data (ω, σ, τ)

Outer action of the mathed pair on the hyperfinite II_1 factor $\mathcal{R} \Longrightarrow$ Galois extension $\mathcal{R}(\alpha, \nu_0) \subseteq \mathcal{R}^{(\beta,G)}$ using classical definition

$a \otimes_B b \longrightarrow ab_0 \otimes b_1$

In this case our formula $a \otimes_B b \longrightarrow a_0 b_0 \otimes \omega^{-1}(a_1, b_1\beta(b_2), S(b_3))b_4$ reduces to $a \otimes_B b \longrightarrow ab_0 \otimes b_1$

Galois extensions and structure theorems for relative Hopf modules

- G group, ω invertible normalized 3-cocycle
  Coquasi-Hopf algebra $H = (\Bbbk G, \omega)$
  A comodule algebra $\Longleftrightarrow$ quasialgebra G-graded (Albuquerque, Majid, 1998-2000)
  $A_e \subseteq A$ is Galois $\Longleftrightarrow$ A is strongly graded

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  $A_e \subseteq A$ is Galois $\Longleftrightarrow$ A is strongly graded
- **Twist invariance:** for τ twist on H, it follows that $B \subseteq A$ *is* H-Galois $\Longleftrightarrow$ $B \subseteq A_{\tau^{-1}}$ is H_τ-Galois

Galois extensions and structure theorems for relative Hopf modules

For Hopf algebras: $can = \varepsilon_{A \otimes H}$, where $A_\bullet \otimes H_\bullet^\bullet \in \mathcal{M}_A^H$

Lemma

S bijective $\Longrightarrow$ isomorphism of right H-comodules $H^\bullet \otimes A^\bullet \longrightarrow A \otimes H^\bullet$

Proposition

$A \otimes H$ becomes a right A-module in $\mathcal{M}^H$
Counit $\varepsilon_{A \otimes H} = can_f$, where f = twist induced by S^{-1}
ε bijective $\Longrightarrow$ Galois extension $B \subseteq A$

Cleft extensions

Definition

A right H-comodule algebra is **cleft** *if* $\exists\ \gamma, \delta : H \longrightarrow A$ such that

$$\rho(\gamma(h)) = \gamma(h_1) \otimes h_1 \qquad \rho(\delta(h)) = \delta(h_2) \otimes S(h_1)$$
$$\delta(h_1)\gamma(h_2) = \alpha(h)1_A \qquad \gamma(h_1)\beta(h_2)\delta(h_3) = \varepsilon(h)1_A$$

Cleft extensions

Proposition

There is a Morita context $\mathbb{M}(A)$ *with*

- *rings* $Hom^H(\overline{H}, A)$ *and* $Hom(H, B)$
- *bimodules* $Hom^H(H, A)$ *and* $Hom^H(H^S, A)$
- *connecting morphisms*

$$\begin{aligned}
(-,-) &: Hom^H(H,A) \otimes_{Hom^H(\overline{H},A)} Hom^H(H^S,A) \longrightarrow Hom(H,B) \\
(\mathfrak{p},\mathfrak{q})(h) &= \mathfrak{p}(h_1)\beta(h_2)\mathfrak{q}(h_3) \\
[-,-] &: Hom^H(H^S,A) \otimes_{Hom(H,B)} Hom^H(H,A) \longrightarrow Hom^H(\overline{H},A) \\
[\mathfrak{q},\mathfrak{p}](h) &= \mathfrak{q}(h_1)\mathfrak{p}(h_2)
\end{aligned}$$

Cleft extensions

Theorem

H coquasi-Hopf algebra, A right H-comodule algebra $\Longrightarrow$

1. *First Morita map* $[,]$ *is surjective* $\Longleftrightarrow$
   $$\begin{cases} B \subseteq A \text{ is Galois} \\ \exists\ n > 0, \text{ s. t. } A \text{ is direct summand in } ({}_\bullet B \otimes H^\bullet)^n \end{cases}$$
2. *Strict Morita context* $\Longleftrightarrow$
   $$\begin{cases} \bullet\ B \subseteq A \text{ is Galois} \\ \bullet\ \exists\ n > 0, \text{ s. t. } A \text{ is direct summand in } ({}_\bullet B \otimes H^\bullet)^n \\ \bullet\ \exists\ n, r > 0 \text{ s. t. } A \text{ is direct summand in } ({}_\bullet B \otimes H^\bullet)^n \text{ and} \\ \qquad {}_\bullet B \otimes H^\bullet \text{ is direct summand in } A^r \end{cases}$$
3. $B \subseteq A$ *cleft extension* $\Longrightarrow$ *strict Morita context.*

Cleft extensions

Theorem

H coquasi-Hopf algebra with bijective antipode, A comodule algebra, $B = A^{coH}$. TFAE:

1. $B \subseteq A$ *cleft*
2. ε_M *bijective for all* $M \in \mathcal{M}_A^H$ *and* $B \subseteq A$ *has the normal basis property*
3. $B \subseteq A$ *Galois and* $B \subseteq A$ *has the normal basis property.*

Then $\mathcal{M}_B \simeq \mathcal{M}_A^H$

Galois extensions and structure theorems for relative Hopf modules

Theorem

H coquasi-Hopf algebra with bijective antipode, A comodule algebra and $B = A^{coH}$. TFAE:

1. *$\exists$ a total integral $\gamma : H \longrightarrow A$ (comodule map with $\gamma(1_H) = 1_A$) and $can : A \otimes_B A \longrightarrow A \otimes H$ is surjective*
2. *The coinvariants functor $(-)^{coH}$ and the induction functor $- \otimes_B A$ form an equivalence of categories $\mathcal{M}_A^H \simeq \mathcal{M}_B$*
3. *(Left version) The coinvariants functor $(-)^{coH}$ and the induction functor $A \otimes_B -$ form an equivalence of categories ${}_A\mathcal{M}^H \simeq {}_B\mathcal{M}$*
4. *A is left B-module faithfully flat and $B \subseteq A$ is Galois*
5. *A is right B-module faithfully flat and $B \subseteq A$ is Galois*

Galois extensions and a Hopf algebroid construction

- H coquasi-Hopf algebra, A right H-comodule algebra
- $A^\bullet \otimes A^\bullet$ is a right comodule by the codiagonal coaction $\rho(a \otimes b) = a_0 \otimes b_0 \otimes a_1 b_1$
- The space of coinvariants $L = (A \otimes A)^{coH}$ is an associative B^{op}-algebra with unit $1_A \otimes 1_A$ and multiplication

$$(a \otimes b)(c \otimes d) = a_0 c_0 \otimes d_0 b_0 \omega^{-1}(a_1, c_1, d_1 b_1)\omega(c_2, d_2, b_2)$$

Galois extensions and a Hopf algebroid construction

- H coquasi-Hopf algebra with bijective antipode, A comodule algebra which is Galois left faithfully flat B-module
  Category equivalence $\mathcal{M}_A^{\mathcal{H}} \simeq \mathcal{M}_B$
- $\mathcal{M}_A^{\mathcal{H}}$ has a natural left $\mathcal{M}^{\mathcal{H}}$-action: $\Diamond : \mathcal{M}^{\mathcal{H}} \times \mathcal{M}_A^{\mathcal{H}} \longrightarrow \mathcal{M}_A^{\mathcal{H}}$, $V \Diamond M = V \otimes M$, with structures $\rho(v \otimes m) = v_0 \otimes m_0 \otimes v_1 m_1$ and $(v \otimes m)a = v \otimes ma$
- It follows that $\mathcal{M}_B$ is a left $\mathcal{M}^{\mathcal{H}}$-category, with structure:

$$\Diamond : \mathcal{M}^{\mathcal{H}} \times \mathcal{M}_B \longrightarrow \mathcal{M}_B \qquad V \Diamond N = [V \otimes (N \otimes_B A_\bullet)]^{co\mathcal{H}}$$

and coassociator $V \Diamond (W \Diamond N)) \xrightarrow{\Psi^{-1}_{V,W,M}} (V \otimes W) \Diamond N$,
$\Psi^{-1}_{V,W,M}(v \otimes \{[w \otimes (n \otimes_B a)] \otimes_B b\}) = (v \otimes w) \otimes (n \otimes_B ab)$

Galois extensions and a Hopf algebroid construction

- A algebra in $\mathcal{M}^{\mathcal{H}} \Longrightarrow$ left A-modules within $\mathcal{M}_A^{\mathcal{H}}$ and $\mathcal{M}_B$
- Obtain equivalent categories ${}_A\mathcal{M}_A^{\mathcal{H}}$ and ${}_A(\mathcal{M}_B)$
- Recover our algebra $L = A\Diamond B^{op}$ and have category isomorphism ${}_A(\mathcal{M}_B) \simeq {}_{A\Diamond B^{op}}\mathcal{M}$ (Hopf algebra case: Schauenburg, 2003)

Proposition

${}_L\mathcal{M}$ is a monoidal category and L becomes a B^{op}-Hopf algebroid.

Further step in constructing Hopf algebroids

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- Then A is a commutative algebra in the category of Yetter-Drinfeld modules

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- Then A is a commutative algebra in the category of Yetter-Drinfeld modules
- Some smash product of A with H should be a Hopf algebroid (for Hopf algebras - Militaru, Brzèzinski, 2001)