On 2-categorical aspects of Hopf algebras

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Hopf algebras, monoidal categories and related topics
July 2022

Problem

- Given a (co)quasi-Hopf algebra, associate a universal Hopf algebra to it.
- Given a coquasi-bialgebra, associate a universal bialgebra to it.
- Universal: left 2-adjoints to the forgetful 2-functors

$$\mathsf{Bialg} \to \mathsf{cqBialg} \qquad \mathsf{HopfAlg} \to \mathsf{cqHopfAlg}$$

Monoids in monoidal categories

- $(\mathcal{V}, \otimes, I)$ locally presentable symmetric monoidal closed category (main example: $\mathsf{Vect}_{\Bbbk}$)
- $\mathsf{Mon}(\mathcal{V})$ category of monoids and monoid morphisms in $\mathcal{V}$: finitarily monadic over $\mathcal{V}$, locally presentable, symmetric monoidal
- $\mathsf{Mon}(\mathcal{V})$ is the 2-category of one-object enriched $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations (2-cells in $\mathsf{Mon}(\mathcal{V})$ are also known as intertwiners)
- $\mathsf{Mon}(\mathcal{V})$ has all conical 2-limits, but lacks other usual 2-limits
- The embedding $\mathsf{Mon}(\mathcal{V}) \hookrightarrow \mathcal{V}\text{-}\mathsf{Cat}$ is strict monoidal and reflective, but not 2-reflective (reflection provided by the pushout in $\mathcal{V}\text{-}\mathsf{Cat}$ of $\mathcal{A} \leftarrow \mathrm{ob}(\mathcal{A}) \to \mathbb{1}$)

Comonoids in monoidal categories

- $\mathrm{Comon}(\mathcal{V}) = (\mathcal{V}^{op}\text{-Mon})^{op}$ 2-category of comonoids and comonoid morphisms in $\mathcal{V}$
- $\mathrm{Comon}(\mathcal{V})$ as an ordinary category: comonadic over $\mathcal{V}$, locally presentable, symmetric monoidal closed
- $\mathrm{Comon}(\mathcal{V})$ as a symmetric monoidal 2-category:

a 2-cell $A \underset{g}{\overset{f}{\Downarrow\alpha}} B$ is $\alpha : A \to I$, $(\alpha \otimes f)\Delta = (g \otimes \alpha)\Delta$

E.g. if A is a (coquasi)bialgebra, then a 2-cell in $\mathrm{Comon}(\mathcal{V})$

$A \underset{1_A}{\overset{u\epsilon}{\Downarrow}} A$ is precisely a left integral of A.

Bimonoids and two-dimensional monad theory

- Monoids in Comon($\mathcal{V}$): bimonoids (equivalently, T-algebras for the free monoid monad $TX = \coprod_{n\geq 0} X^{\otimes n}$ on Comon($\mathcal{V}$))
- T is in fact a 2-monad on the 2-category Comon($\mathcal{V}$)
- Hence strict/pseudo/(co)lax T-algebras and strict/pseudo/(co)lax T-morphisms are available

$$
\begin{array}{ccccc}
T\text{-}\mathrm{Alg}_{\mathrm{strict}} & \longrightarrow & T\text{-}\mathrm{Alg}_{\mathrm{pseudo}} & \longrightarrow & \mathrm{Ps}\text{-}T\text{-}\mathrm{Alg} \\
& \searrow & \downarrow & \swarrow & \\
& & \mathrm{Comon}(\mathcal{V}) & &
\end{array}
$$

Bimonoids and two-dimensional monad theory

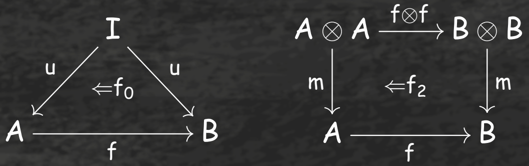
- The 2-category of strict T-algebras and strict T-morphisms: the familiar Bimon($\mathcal{V}$) (bimonoids and bimonoid morphisms)

$$\text{a 2-cell is } A \underset{g}{\overset{f}{\Downarrow\alpha}} B \text{ is } \quad \alpha : A \to I \quad \begin{cases} (\alpha \otimes f)\Delta = (g \otimes \alpha)\Delta \\ \alpha u = u \\ \alpha m = m(\alpha \otimes \alpha) \end{cases}$$

- Bimon($\mathcal{V}$) locally presentable, with zero object, enriched in Comon($\mathcal{V}$) via Bimon($\mathcal{V}$) $\to$ Mon($\mathcal{V}$)

Bimonoids and two-dimensional monad theory

- The 2-category of strict T-algebras and pseudo-T-morphisms: $\mathrm{Bimon}(\mathcal{V})_{\mathrm{ps}}$ (unit and multiplication preserved up to coherent iso-2-cells $f_0 : I \to I$ scalar, $f_2 : A \otimes A \to I$ cocycle)



- Any pseudo-morphism can be strictified: factorise it as the identity "on objects" followed by a strict morphism

$$A \xrightarrow{(f,f_2)} B \qquad = \qquad A \xrightarrow{(1,f_2)} A_{f_2} \xrightarrow{(f,1)} B$$

- Similarity with the (bo,ff) factorisation system on Cat

Bimonoids and two-dimensional monad theory

- For a 2-monad T with rank on a complete and cocomplete 2-category, $T-\mathrm{Alg}_{\mathrm{strict}} \to T-\mathrm{Alg}_{\mathrm{pseudo}}$ has left adjoint.
- But $\mathrm{Comon}(\mathcal{V})$ fails to be (at least!) cocomplete
- Algebras for ordinary monads are reflexive coequalizers of free ones
- The corresponding 2-categorical notion: reflexive coherence data

$$(T^3A,\mu T^2)\ \begin{array}{c} \xrightarrow{\quad \mu T \quad} \\ \xleftarrow{\quad T^2\eta \quad} \\ \xrightarrow{\quad T\mu \quad} \\ \xleftarrow{\quad T\eta T \quad} \\ \xrightarrow{\quad T^2 a \quad} \end{array}\ (T^2A,\mu T)\ \begin{array}{c} \xrightarrow{\quad \mu \quad} \\ \xleftarrow{\quad T\eta \quad} \\ \xrightarrow{\quad Ta \quad} \end{array}\ (TA,\mu)$$

- If T is a 2-monad for which $T-\mathrm{Alg}_{\mathrm{strict}}$ admits codescent objects, then $T-\mathrm{Alg}_{\mathrm{strict}} \to T-\mathrm{Alg}_{\mathrm{pseudo}}$ has left adjoint

Coquasi-bialgebras

- cQBimon($\mathcal{V}$): 2-category of pseudo T-algebras and pseudo T-morphisms

Objects: pseudo-monoids $(A, u : I \to A, m : A \otimes A \to A)$ in the category of comonoids (e.g. coquasi-bialgebras)

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes 1} & A \otimes A \\
{\scriptstyle 1 \otimes m}\downarrow & \cong & \downarrow{\scriptstyle m} \\
A \otimes A & \xrightarrow[m]{} & A
\end{array}
\qquad
\begin{array}{ccccc}
A & \xrightarrow{u \otimes 1} & A \otimes A & \xleftarrow{1 \otimes u} & A \\
 & \searrow \ \cong & \downarrow{\scriptstyle m} & \cong \ \swarrow & \\
 & & A & &
\end{array}
$$

1-cells: $A \overset{(f, f_2, f_0)}{\to} B$

$$
\begin{array}{ccc}
 & I & \\
{\scriptstyle u}\swarrow & \Leftarrow f_0 & \searrow{\scriptstyle u} \\
A & \xrightarrow[f]{} & B
\end{array}
\qquad
\begin{array}{ccc}
A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\
{\scriptstyle m}\downarrow & \Leftarrow f_2 & \downarrow{\scriptstyle m} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

2-cells: ...

Coquasi-bialgebras

- No known results on limits and colimits of cQBimon($\mathcal{V}$)
- If T is a 2-monad for which $T-Alg_{strict}$ admits codescent objects of reflexive coherence data, then $T-Alg_{strict} \to Ps-T-Alg$ has left adjoint.
- If T is a 2-monad on a 2-category endowed with an enhanced factorisation system (E, M) such that if $m \in M$ and $mf \cong 1$ then $fm \cong 1$, and T preserves the E class, then every pseudo T-algebra is equivalent to a strict one
- However, this cannot happen (e.g. H(2))
- Any ideas?

Possible directions

- Strictification of comodules + Tannaka reconstruction
- Bicategory of bicomodules Comod($\mathcal{V}$) (advantages: richer structure, locally complete and cocomplete, has all right liftings and extensions, coquasi-bialgebras are map pseudomonoids)