On 2-categorical aspects of (quasi)bialgebras

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Geometric Methods in Mathematics
TU Dresden
January 2024

Problems

- Given a (co)quasi-Hopf algebra, associate a universal Hopf algebra to it.
- Given a coquasi-bialgebra, associate a universal bialgebra to it.
- Universal: left adjoints to the forgetful functors

$$\mathsf{Bialg} \to \mathsf{cqBialg} \qquad \mathsf{HopfAlg} \to \mathsf{cqHopfAlg}$$

- Find the appropriate categorical setting: 1-categories, 2-categories, bicategories, double categories ...

Monoids in monoidal categories

- $(\mathcal{V}, \otimes, I)$ locally presentable symmetric monoidal closed category (main example: $\mathsf{Vect}_{\mathbb{k}}$)

- $\mathcal{V}$-Mon category of monoids and monoid morphisms in $\mathcal{V}$: finitarily monadic over $\mathcal{V}$, locally presentable, symmetric monoidal

- $\mathcal{V}$-Mon is the 2-category of one-object enriched $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations (2-cells in $\mathcal{V}$-Mon are also known as intertwiners)

- The embedding $\mathcal{V}\text{-Mon} \hookrightarrow \mathcal{V}\text{-Cat}$ is strict monoidal and reflective, but not 2-reflective (reflection provided by the pushout in $\mathcal{V}$-Cat of $\mathcal{A} \leftarrow \mathrm{ob}(\mathcal{A}) \to \mathbb{1}$)

Monoids in monoidal categories

- $\mathcal{V}$-Mon has all conical 2-limits, but lacks other usual 2-limits: cotensors, inserters, equifiers, ...

- Coequalizers in $\mathcal{V}$-Mon are 2-categorical, but coproducts no; however, $\mathcal{V}$-Mon has coinserters and coequifiers

Coinserter

$$A \overset{f}{\underset{g}{\rightrightarrows}} B \longrightarrow B < x > / < xf(a) - g(a)x, \forall a \in A >$$

(in $\mathrm{Vect}_{\Bbbk}$)

Coequifier

$$A \; \overset{f}{\underset{g}{\rightrightarrows}} \; (\alpha \Downarrow \quad \Downarrow \beta) \; B \longrightarrow B / < \alpha - \beta >$$

(in $\mathrm{Vect}_{\Bbbk}$)

Comonoids in monoidal categories

- $\mathcal{V}$-Comon = $(\mathcal{V}^{op}\text{-Mon})^{op}$ 2-category of comonoids and comonoid morphisms in $\mathcal{V}$

- $\mathcal{V}$-Comon as an ordinary category: comonadic over $\mathcal{V}$, locally presentable, symmetric monoidal closed

- $\mathcal{V}$-Comon as a symmetric monoidal 2-category:

$$\text{a 2-cell } A \underset{g}{\overset{f}{\Downarrow \alpha}} B \text{ is } \quad \alpha : A \to I, \quad (\alpha \otimes f)\Delta = (g \otimes \alpha)\Delta$$

  E.g. if A is a (coquasi)bialgebra, then a 2-cell in $\mathcal{V}$-Comon $A \underset{1_A}{\overset{u\epsilon}{\Downarrow}} A$

  is precisely a left integral of A.

- 2-limits and 2-colimits: dualise the results for $\mathcal{V}$-Mon

Bimonoids and two-dimensional monad theory

- Monoids in $\mathcal{V}$-Comon: bimonoids (equivalently, T-algebras for the free monoid monad $TX = \coprod_{n \geq 0} X^{\otimes n}$ on $\mathcal{V}$-Comon)
- T is in fact a 2-monad on the 2-category $\mathcal{V}$-Comon
- Hence strict/pseudo/(co)lax T-algebras and strict/pseudo/ (co)lax T-morphisms are available

$$\begin{array}{ccccc} T\text{-}\mathrm{Alg}_{\mathrm{strict}} & \longrightarrow & T\text{-}\mathrm{Alg}_{\mathrm{pseudo}} & \longrightarrow & \mathrm{Ps}\text{-}T\text{-}\mathrm{Alg} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{V}\text{-Comon} & & \end{array}$$

Bimonoids and two-dimensional monad theory

- The 2-category of strict T-algebras and strict T-morphisms

$$T\text{-}\mathrm{Alg}_{\mathrm{strict}} = \mathcal{V}\text{-Bimon}$$

(the familiar (2-)category of bimonoids and bimonoid morphisms)

a 2-cell is $A \underset{g}{\overset{f}{\Downarrow \alpha}} B$ is $\alpha : A \to I$ $\begin{cases} (\alpha \otimes f)\Delta = (g \otimes \alpha)\Delta \\ \alpha u = u \\ \alpha m = m(\alpha \otimes \alpha) \end{cases}$

- [Porst 2008] $\mathcal{V}$-Bimon as ordinary category: locally presentable, monadic over $\mathcal{V}$-Comon, with zero object

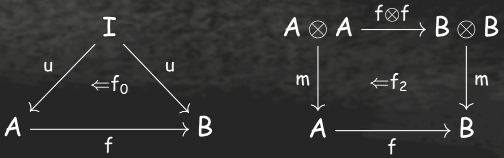
Bimonoids and two-dimensional monad theory

- The 2-category of strict T-algebras and pseudo-T-morphisms:

$$\mathcal{V}\text{-Bimon}_{ps}$$

- Objects: $\mathcal{V}$-bimonoids again

- Pseudo-T-morphisms: comonoid morphisms which preserve unit and multiplication up to coherent iso-2-cells $f_0 : I \to I$ scalar, $f_2 : A \otimes A \to I$ (twist cocycle)



Bimonoids and two-dimensional monad theory

- [Blackwell-Kelly-Power 1989] For a 2-monad T with rank on a complete and cocomplete 2-category, the embedding

$$\mathsf{T}-\mathsf{Alg}_{\mathsf{strict}} \to \mathsf{T}-\mathsf{Alg}_{\mathsf{pseudo}}$$

  has left adjoint.

- But $\mathcal{V}$-Comon fails to be 2-cocomplete, although the free monoid monad is (at least!) finitary as an ordinary functor.

- Need another approach.

Bimonoids and two-dimensional monad theory

- Algebras for ordinary monads are reflexive coequalizers of free ones

$$(T^2A, \mu T) \underset{Ta}{\overset{\mu}{\underset{\longleftarrow T\eta \;}{\rightrightarrows}}} (TA, \mu) \longrightarrow A$$

- The corresponding 2-categorical notion: reflexive codescent data

$$(T^3A, \mu T^2) \; \begin{matrix} \xrightarrow{\mu T} \\ \xleftarrow{T^2\eta} \\ \xrightarrow{T\mu} \\ \xleftarrow{T\eta T} \\ \xrightarrow{T^2 a} \end{matrix} \; (T^2A, \mu T) \; \begin{matrix} \xrightarrow{\mu} \\ \xleftarrow{T\eta} \\ \xrightarrow{Ta} \end{matrix} \; (TA, \mu) \longrightarrow A$$

- [Lack 2002] If T is a 2-monad for which $T-Alg_{strict}$ admits codescent objects, then

$$T-Alg_{strict} \to T-Alg_{pseudo}$$

  has left adjoint.

- This is the case in particular if $T-Alg_{strict}$ admits coinserters and coequifiers.

Bimonoids and two-dimensional monad theory

- But $\mathcal{V}$-Mon has coinserters and coequifiers for any $\mathcal{V}$; in particular, substitute $\mathcal{V}$ by $\mathcal{V}$-Comon

- Theorem The embedding

$$\mathcal{V}\text{-Bimon} \to \mathcal{V}\text{-Bimon}_{\mathrm{ps}}$$

  has a left adjoint.

- That is, any pseudo-morphism can be strictified: factorise it as the identity "on objects" followed by a strict morphism

$$A \xrightarrow{(f,f_2)} B \qquad = \qquad A \xrightarrow{(1,f_2)} A_{f_2} \xrightarrow{(f,1)} B$$

- Similarity with the (bo,ff) factorisation system on Cat

Coquasi-bimonoids

- $\mathcal{V}$-cQBimon = Ps − T − Alg: the 2-category of pseudo-T-algebras and pseudo-T-morphisms

- Objects: pseudo-monoids $(A, u : I \to A, m : A \otimes A \to A)$ in the category of comonoids (e.g. coquasi-bimonoids)

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes 1} & A \otimes A \\ {\scriptstyle 1 \otimes m}\downarrow & \cong & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow[m]{} & A \end{array} \qquad \begin{array}{ccccc} A & \xrightarrow{u \otimes 1} & A \otimes A & \xleftarrow{1 \otimes u} & A \\ & \searrow \;\cong & \downarrow{\scriptstyle m} & \cong\; \swarrow & \\ & & A & & \end{array}$$

- 1-cells: $A \overset{(f, f_2, f_0)}{\to} B$

$$\begin{array}{ccc} & I & \\ {\scriptstyle u}\swarrow & \Leftarrow f_0 & \searrow{\scriptstyle u} \\ A & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \\ {\scriptstyle m}\downarrow & \Leftarrow f_2 & \downarrow{\scriptstyle m} \\ A & \xrightarrow[f]{} & B \end{array}$$

- 2-cells: ...

Coquasi-bimonoids: coherence and strictification

- [Power 1989] If T is a 2-monad on a 2-category endowed with an enhanced factorisation system (E, M) such that if $m \in M$ and $mf \cong 1$ then $fm \cong 1$, and T preserves the E class, then every pseudo T-algebra is equivalent to a strict one.

- However, this cannot happen here (take for example the quasi-Hopf algebra $H(2)$ which is not twist equivalent to any Hopf algebra)

- [Lack 2002] If T is a 2-monad for which $T-\mathrm{Alg}_{\mathrm{strict}}$ admits codescent objects, then also

$$T-\mathrm{Alg}_{\mathrm{strict}} \to \mathrm{Ps}-T-\mathrm{Alg}$$

  has a left adjoint.

- Consequence: there exists a left adjoint to the embedding

$$\mathcal{V}\text{-Bimon} \to \mathcal{V}\text{-cQBimon}$$

  but the components of the unit cannot always be equivalences in $\mathcal{V}$-cQBimon.

Work in progress

- Description of the left adjoint.
- Limits and colimits of $\mathcal{V}$-cQBimon.
- Here only the 2-categorical machinery was used.
- In fact, $\mathcal{V}$-Comon has a righter structure: it is a fibrant double category, and all the subsequent constructions translate into this context.
- In particular, the associated horizontal bicategory of $\mathcal{V}$-comonoids and bicomodules is locally complete and cocomplete and has all right liftings and extensions.
- Also, the notion of (coquasi-)Hopf $\mathcal{V}$-monoid fits naturally, as a left autonomous map pseudomonoid in $\mathcal{V}$-Comon [Lopez-Franco 2009].