

Finitary Functors: from **Set** to **Preord** and **Poset**

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Abstract. We investigate how finitary functors on **Set** can be extended or lifted to finitary functors on **Preord** and **Poset** and discuss applications to coalgebra.

Keywords: extension; lifting; relator; simulation; (final) coalgebra; exact square; embedding

1 Introduction

Endofunctors $T : \mathbf{Set} \rightarrow \mathbf{Set}$ play a crucial role in the theory of coalgebras and the rich body of results on them [4] has been exploited over the years to prove results about the category $\mathbf{Coalg}(T)$, and about logics for T -coalgebras, uniformly in the functor T .

Not as dominant as **Set**-functors, functors on preorders and on posets have made their appearance, for example, if one is interested in simulation rather than only bisimulation [8,12]. Moreover, we think of the categories **Preord** and **Poset** as the natural link between universal coalgebra [14] and domain theory [1], as domains are special posets.

A general plan of work would be the comprehensive study of **Preord**- and **Poset**-functors and their relationship to **Set**-functors and to coalgebras. In this paper, we restrict ourselves to the modest approach of transforming **Set**-functors into **Preord** and **Poset**-functors and study how some properties important from the coalgebraic point of view are transferred. Two notions arise here: extension and lifting of a **Set**-functor T , where extension means a functor which coincides with T on discrete set and lifting means that underlying sets are kept but some order is added.

For extensions, the final coalgebra is discrete, but nevertheless the associated notion of (bi)simulation on posets can be interesting. For liftings, the order on the final coalgebra is similarity, an insight going back to [13, Thm 4.1] and [17, Thm 5.9].

We start from the observation that every finitary **Set**-functor T has a (canonical) presentation as a coequalizer of two polynomial functors. Since sets are discrete preorders, this coequalizer can be computed in preorders (or posets) to yield a functor $\tilde{T} : \mathbf{Preord} \rightarrow \mathbf{Preord}$, which simultaneously lifts and extends T . As shown in [16] this leads to interesting examples: if T is the finite powerset

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functor, then \tilde{T} on Poset yields the finite(ly generated) convex powerset functor. On the other hand, the final \tilde{T} -coalgebra is always discrete and, therefore, does not capture a notion of simulation.

So we study quotients of polynomial functors, but now with *ordered coefficients* and show that this is equivalent to the notion of an order $\bar{T} : \text{Set} \rightarrow \text{Preord}$ on a functor T from [8]. The latter investigate conditions under which \bar{T} can be lifted to a functor $\hat{T} : \text{Preord} \rightarrow \text{Preord}$. On the other hand, again interpreting the coequalizer of polynomial functors in Preord , we obtain another lifting $\check{T} : \text{Preord} \rightarrow \text{Preord}$, which always exists. We show $\hat{T} = \check{T}$ under the conditions which ensure the existence of \hat{T} . The table below summarizes the notation for the various extensions and liftings met in the paper.

$T : \text{Set} \rightarrow \text{Set}$
$\bar{T} : \text{Set} \rightarrow \text{Preord}$ (Def. 9)
$\tilde{T} : \text{Preord} \rightarrow \text{Preord}$ Extension (3.2)
$\hat{T}, \check{T} : \text{Preord} \rightarrow \text{Preord}$ Liftings (Def.-Prop. 16, resp. Def. 24)

The last section of the paper focuses on Poset -functors obtained from the previous constructions by taking quotients, with similar results obtained.

Finally, further topics pursued in the paper are the preservation of exact squares (the ordered analogue of weak pullbacks) and of embeddings. The latter is motivated by the result on the expressiveness of modal logic over posets [9], while the former comes from the fact that it replaces preservation of weak pullbacks as a condition for the existence of the relation lifting on preorders or posets [5].

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2 Preliminaries

We denote by Preord and Poset the categories of preordered sets and of posets, respectively, and monotone maps. We write $D \dashv U : \text{Preord} \rightarrow \text{Set}$ for the adjunction between the discrete and the forgetful functor. As U has also a right adjoint (which endows a set with the indiscrete preorder), it preserves all limits and colimits. In particular, coequalizers in Preord are computed as in Set , namely for any pair of monotone maps

$$(X, \leq) \rightrightarrows (Y, \leq) \xrightarrow{\pi} Z ,$$

their Set -coequalizer Z , with the smallest preorder such that π is monotone, becomes the coequalizer in Preord . We denote by $Q \dashv J : \text{Poset} \rightarrow \text{Preord}$ the adjunction between the quotient functor (sending every preordered set to the quotient poset obtained by identifying all x, y with $x \leq y$ and $y \leq x$) and the

inclusion functor. For later use, recall that coequalizers in **Poset** are computed in two steps: first, take the coequalizer in **Preord**, then quotient it to obtain a poset.

An *embedding* in **Preord** or **Poset** is an injective, monotone and order reflecting map. In both categories, the embeddings are precisely the strong monomorphisms. An *exact square* [7] in **Preord** or in **Poset** is a diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & X \\ \beta \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array} \quad (2.1)$$

with $f\alpha \leq g\beta$, such that

$$\forall x \in X, y \in Y. f(x) \leq g(y) \Rightarrow \exists p \in P. x \leq \alpha(p) \wedge \beta(p) \leq y. \quad (2.2)$$

If P is $\{(x, y) \in X \times Y \mid f(x) \leq g(y)\}$ with the product order and α and β are the usual projections, then (2.2) is obviously satisfied. (2.1) is then called a comma square.

The terminology is borrowed from [7], where exact squares were introduced in the framework of 2-categories. See also [11], where equalizers were similarly replaced by subequalizers. In **Set**, an exact square is precisely a weak pullback. In [12], a commutative square having the property (2.2) is called a preorder quasi-pullback.

Let T be a **Set**-functor. It is well-known that T is finitary (commutes with ω -filtered colimits) if and only if it admits a coend representation

$$TX \cong \int^{n < \omega} \text{Set}(n, X) \bullet T\underline{n},$$

that is, TX has a presentation given by the coequalizer

$$\coprod_{m, n < \omega} \text{Set}(\underline{m}, \underline{n}) \times T\underline{m} \times X^n \begin{array}{c} \xrightarrow{\rho_X} \\ \xrightarrow{\lambda_X} \end{array} \coprod_{n < \omega} T\underline{n} \times X^n \xrightarrow{\pi_X} TX, \quad (2.3)$$

where \underline{n} refers to $\{0, 1, \dots, n-1\}$, and the pair (λ_X, ρ_X) is given by $\lambda_X(f, \sigma, x) = (Tf(\sigma), x)$ and $\rho_X(f, \sigma, x) = (\sigma, x \circ f)$, for $f : \underline{m} \rightarrow \underline{n}$, $x : \underline{n} \rightarrow X$ and $\sigma \in T\underline{m}$.³ Also, $\pi_X(\sigma, x) = Tx(\sigma)$. Intuitively, $T\underline{n}$ can be seen as the set of operations of arity n applied to the variables in X . In the sequel, we shall omit the subscript X when referring to the maps λ_X, ρ_X and π_X if the context is clear.

3 From Set to Preord

Section 3.1 considers the notions of extensions and liftings. Section 3.2 introduces the preordification \tilde{T} of a **Set**-functor T based on the presentation of T .

³ We shall identify functions $x : \underline{n} \rightarrow X$ with tuples $x = (x_0, \dots, x_{n-1})$, where $x_i = x(i), \forall i \in \{0, \dots, n-1\}$.

Section 3.3 shows that putting an order on the coefficients of the presentation agrees with the notion of an order \bar{T} on a **Set**-functor T from [8]. Section 3.4 recalls how [8] use relation lifting to extend \bar{T} to an endofunctor \hat{T} on **Preord** and shows that if T preserves weak pullbacks then \hat{T} preserves exact squares and, therefore, embeddings. Section 3.5 emphasizes that the order on the final \hat{T} -coalgebra coincides with the similarity given by relation lifting. Whereas [8] use relation lifting to extend \bar{T} to a functor \hat{T} on preorders, we can now also use the presentation to extend \bar{T} to a functor \check{T} on preorders. Section 3.6 shows that $\check{T} = \hat{T}$ under the conditions given in [8] for the existence of \hat{T} .

3.1 Extension and Lifting

Definition 1. *Let T be a **Set**-functor. An extension of T to **Preord** is a locally monotone functor $\Gamma : \mathbf{Preord} \rightarrow \mathbf{Preord}$ such that $\Gamma D = DT$. A lifting of T to **Preord** is a locally monotone functor $\Gamma : \mathbf{Preord} \rightarrow \mathbf{Preord}$ such that $U\Gamma = TU$.*

In the following, if T is finitary, we also require Γ (extension or lifting) to be so.⁴ Extensions and liftings of a **Set**-functor to **Poset** are defined similarly.

It follows that both a lifting and an extension will satisfy the relation $T = UFD$. Intuitively, an extension will coincide with T on discrete sets, while a lifting means that we put a preorder (respectively a partial order) on TX . Also, there is an immediate test to decide whether a (finitary) locally monotone **Preord** (or **Poset**)-functor Γ is a lifting or an extension of a **Set**-functor: namely, compute $T = UFD$ and check if $\Gamma D = DT$ or $U\Gamma = TU$.

Remark 2. An extension is not necessarily unique. Let Γ be the functor sending a preordered set (a poset) to the (discrete) set of its connected components. Then $U\Gamma D = Id$. But also the identity on **Preord** (respectively on **Poset**) is an extension of Id , showing that a **Set**-functor can have different extensions.

The local monotonicity requirement is natural, as the categories **Preord** and **Poset** are enriched over themselves (in the sense that the hom-sets are ordered) and enriched functors coincide with locally monotone ones. In all constructions that we shall perform, the local monotonicity of the lifted/extended functor will come for free.

Note that although **Set** is (discretely) enriched over **Preord** (and over **Poset**), the adjunction $D \dashv U$ is not, since U is not locally monotone. In particular, $\Gamma = DTU$ will not in general be an extension/lifting and we shall replace it by (3.2) instead.

⁴ An extension Γ of a finitary functor T need not be finitary: consider the finitary functor $TX = \{l : \mathbb{N} \rightarrow X \mid l(n) = l(n+1) \text{ for all but a finite number of } n\}$. It admits the **Preord**-extension $\Gamma(X, \leq_X) = \{l : (\mathbb{N}, \leq_{\mathbb{N}}) \rightarrow (X, \leq_X) \mid l(n) \leq l(n+1) \text{ for all but a finite number of } n\}$. But this Γ is not finitary: take the family of finite sets (\underline{n}, \leq) ordered as usual, with inclusion maps, whose colimit is $(\mathbb{N}, \leq_{\mathbb{N}})$. Then one can check that $\text{colim} \Gamma(\underline{n}, \leq) \not\cong \Gamma(\text{colim}(\underline{n}, \leq))$. We would like to thank the anonymous referee for pointing us this example. We didn't succeed in constructing a similar example for liftings.

If T is a (finitary) **Set**-functor and Γ is an extension of T , then T -coalgebras and Γ -coalgebras are related by an adjunction $\tilde{C} \dashv \tilde{D} : \mathbf{Coalg}(T) \rightarrow \mathbf{Coalg}(\Gamma)$ which can be easily derived from the adjunction $C \dashv D : \mathbf{Set} \rightarrow \mathbf{Preord}$ between the connected components functor and the discrete functor, using that $T = C\Gamma D$ if Γ is an extension of T .⁵ Consequently, \tilde{D} will preserve limits, in particular, the final coalgebra (if it exists).

Proposition 3. *For any (finitary) extension Γ of a finitary functor T , the final Γ -coalgebra does exist and is discrete.*

The situation slightly changes when we consider a lifting instead of an extension. In this case there is an obvious forgetful functor $\tilde{U} : \mathbf{Coalg}(\Gamma) \rightarrow \mathbf{Coalg}(T)$, which has a left adjoint $\tilde{D} : \mathbf{Coalg}(T) \rightarrow \mathbf{Coalg}(\Gamma)$ (this is not hard to check). Thus \tilde{U} preserves limits; in particular the underlying set of the final Γ -coalgebra (if it exists) will be the final T -coalgebra.

Proposition 4. *For any (finitary) lifting Γ of a finitary functor T , the final Γ -coalgebra exists [10] and is built on the same set as the final T -coalgebra.*

3.2 First Construction: Order on the Variables

Going back to (2.3), and following [16], we are now interested in this coequalizer if we replace the set X by a preorder (X, \leq) . The other sets involved in (2.3) remain discretely ordered, except for X^n , which carries the product order from (X, \leq) . Then λ, ρ are monotone. The coequalizer in **Preord** of this monotone pair of maps (λ, ρ) has the same underlying set TX , but now with a preorder \trianglelefteq :

$$\coprod_{m, n < \omega} \mathbf{Set}(m, n) \times Tm \times (X^n, \leq) \xrightarrow[\lambda]{\rho} \coprod_{n < \omega} Tn \times (X^n, \leq) \xrightarrow{\pi} (TX, \trianglelefteq) \quad (3.1)$$

If $f : (X, \leq) \rightarrow (Y, \leq)$ is a monotone map, it follows that $Tf : (TX, \trianglelefteq) \rightarrow (TY, \trianglelefteq)$ is monotone. Thus we obtain a functor which is also locally monotone

$$\tilde{T} : \mathbf{Preord} \rightarrow \mathbf{Preord}, \quad \tilde{T}(X, \leq) = (TX, \trianglelefteq) \quad (3.2)$$

It simultaneously defines an extension and a lifting of T . In fact, it is an enriched coend $\tilde{T}(X, \leq) \cong \int^{n < \omega} [Dn, (X, \leq)] \bullet DTn \times$, where $[Dn, (X, \leq)]$ refers to the preordered set (internal hom) of all monotone maps from Dn to X .

A functor may have different presentations,⁶ but we have

Proposition 5. *\tilde{T} is independent of the chosen presentation of T .*

Example 6. 1. Let $T = \mathcal{P}_f$, the finite powerset functor. For (X, \leq) a preordered set, the above construction leads to the Egli-Milner preorder on $\mathcal{P}_f X$: $u \trianglelefteq v$ for $u, v \subseteq X$ finite iff $\forall a \in u \exists b \in v. a \leq b$ and $\forall b \in v \exists a \in u. a \leq b$.

⁵ We leave the details to the reader. Notice there is a similar duality for **Poset**-extensions.

⁶ For example, the finite powerset functor \mathcal{P}_f can be presented as in (2.3), but also as a quotient of the list-functor $\coprod_{n < \omega} X^n$.

2. Take $TX = 1 + X$ the lift functor. For (X, \leq) a preordered set, the corresponding order \sqsubseteq on $1 + X$ will be the coproduct order.
3. For the list functor $TX = X^*$, a preorder on lists is obtained as follows:

$$[x_0 \dots x_{n-1}] \sqsubseteq [y_0 \dots y_{m-1}] \Leftrightarrow m = n \wedge x_i \leq y_i, \forall i < n.$$

In Sect. 3.4 we will see another description of \sqsubseteq based on relation liftings.

3.3 Second Construction: Order on the Operations

We now equip a functor T with an order on the coefficients of its presentation.

Definition 7. *Let T be a finitary Set-functor. We say that T has a presentation with a preorder, if for each finite arity n , there is a preorder \leq on $T\underline{n}$ such that $Tf : (T\underline{m}, \leq) \rightarrow (T\underline{n}, \leq)$ is monotone for all $f : \underline{m} \rightarrow \underline{n}$.*

There are many functors who carry a natural order, as eg the powerset functor (with the inclusion order), or the lift functor $TX = \{\perp\} + X$, with the flat order $\perp \leq x, \forall x \in X$ (see Example 11). The latter is a special case of the following:

Example 8. Let T be a (finitary) Set-functor, but not the constant functor mapping everything to the empty set. Then $T1 \neq \emptyset$. Specify a preorder on $T1$. This will induce a preorder on $T\underline{n}$ for all $n < \omega$ via the image of the map $\underline{n} \rightarrow 1$ through T , namely the preorder on $T\underline{n}$ is the inverse image of the order on $T1$. Then we obtain an order on T .

Definition 9. *Let T be a finitary functor with preorder \leq . Consider on TX the preorder \sqsubseteq obtained from the coequalizer*

$$\coprod_{m, n < \omega} \text{Set}(\underline{m}, \underline{n}) \times (T\underline{m}, \leq) \times X^n \begin{array}{c} \xrightarrow{\rho} \\ \xrightarrow{\lambda} \end{array} \coprod_{n < \omega} (T\underline{n}, \leq) \times X^n \xrightarrow{\pi} (TX, \sqsubseteq) \quad (3.3)$$

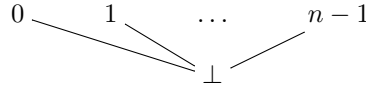
This defines a functor $\bar{T} : \text{Set} \rightarrow \text{Preord}$, $\bar{T}X = (TX, \sqsubseteq)$.

Notice that ρ is always monotone, while λ is so by Def. 7. Therefore it makes sense to compute the above coequalizer in **Preord**. The functor \bar{T} is finitary and satisfies $U\bar{T} = T$. In [8], such a functor is called an order on T . We keep the same terminology. This means that on each TX there is a preorder \sqsubseteq , and these preorders must be preserved by renaming: for each map $f : X \rightarrow Y$, its image $Tf : (TX, \sqsubseteq) \rightarrow (TY, \sqsubseteq)$ is monotone. Choosing \underline{n} for X , we find

Proposition 10. $(T\underline{n}, \sqsubseteq) = (T\underline{n}, \leq)$.

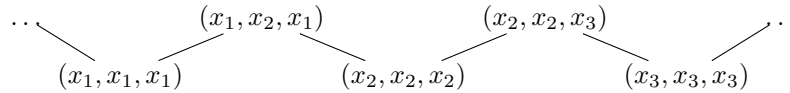
Example 11. 1. Take all $T\underline{n}$ to be discretely ordered. Then T automatically satisfies Def. 7. The preorder obtained on TX will be the discrete one, as any coequalizer of discrete preordered sets is again discrete (D preserves all colimits being left adjoint, in particular coequalizers).

2. Take all $T\underline{n}$ to be indiscretely ordered. If T is finitary then all $(TX, \sqsubseteq) = TX \times TX$ are again indiscrete. Indeed, take any $u, v \in TX$. As T is finitary, we can find a finite set \underline{n} and an injection $x : \underline{n} \rightarrow X$ such that u, v lie in the image of the map $Tx : T\underline{n} \rightarrow TX$. So $u = Tx(\sigma)$ and $v = Tx(\tau)$ with $\sigma, \tau \in T\underline{n}$. As Tx is monotone and σ, τ are comparable, it follows that $u \sqsubseteq v$.
3. Let $T = \mathcal{P}_f$ be the finite powerset functor, with the inclusion order on $\mathcal{P}_f(\underline{n})$. Then the resulting order on any $\mathcal{P}_f X$ is again the inclusion: take any finite subsets $u, v \subseteq X$. Then $u \sqsubseteq v$ if we can find $\sigma \subseteq \underline{n}, x : \underline{n} \rightarrow X, \tau \subseteq \underline{m}, y : \underline{m} \rightarrow X$ such that $\pi(\sigma, x) = u, \pi(\tau, y) = v$, and (σ, x) and (τ, y) are comparable in $\coprod_{n < \omega} \mathcal{P}_f(\underline{n}) \times X^n$. But this can be possible only if both lie in the same component, so $m = n$, and share same variables, $x = y$. It follows $\sigma \subseteq \tau$, hence $u = \mathcal{P}_f(x)(\sigma) \subseteq \mathcal{P}_f(x)(\tau) = v$. Similarly, if on $\mathcal{P}_f(\underline{n})$ we consider the converse inclusion, the resulting preorder \sqsubseteq is \sqsubseteq^{op} .
4. Take now $TX = \{\perp\} + X$. On the associated signature $T\underline{n} = \{\perp\} + \underline{n}$ consider the flat order $\perp < i, \forall i < n$.



The quotient function $\pi : \coprod_{n < \omega} T\underline{n} \times X^n \rightarrow TX$ maps $(\perp, x : n \rightarrow X)$ to \perp and $(i, x : n \rightarrow X)$ to x_i . It follows now easily that \perp will be the least element in $(\{\perp\} + X, \sqsubseteq)$ and that different elements of X are not comparable, hence on TX we get the same flat order.

5. Consider the polynomial functor $TX = \mathbb{N} \times X$. On each $\mathbb{N} \times \underline{n}$, take the preorder given by $(N, i) \leq (M, j) \Leftrightarrow N < M$ or $(N = M$ and $i = j)$. This is precisely the lexicographic order with respect to the usual ordering of \mathbb{N} , when \underline{n} is considered discrete. Then the induced preorder on the quotient TX is similar: two pairs (N, x) and (M, y) in $\mathbb{N} \times X$ are comparable if either are equal or the first components are comparable. In the future, we shall denote this functor by $TX = \mathbb{N} \times X$ to emphasize the special preorder.
6. Let $T = (-)_2^3$ be the functor introduced by Aczel and Mendler in [2], given on objects by $X_2^3 = \{x = (x_1, x_2, x_3) \in X^3 \mid |(x_1, x_2, x_3)| \leq 2\}$. There is a natural (pre)ordering \sqsubseteq on X_2^3 as follows: all triples $(x_1, x_2, x_3) \in X_2^3$ with equal components are minimal elements, all the others are maximals, and a minimal element is comparable with a maximal one only if they share a common component, as in the picture below:



We shall call this the zig-zag preorder. If we restrict it to $T\underline{n}$ and compute (TX, \sqsubseteq) as in (3.3), we obtain again precisely the zig-zag order.

7. Take the list functor $TX = X^*$. Put on each TX the following order

$$[x_0 \dots x_{n-1}] \leq [y_0 \dots y_{m-1}] \Leftrightarrow \exists \varphi : \underline{n} \rightarrow \underline{m} . x_i = y_{\varphi(i)}, \forall i < n ,$$

see [8, Example 2.2.(3)], where the function φ is required to be strictly monotone. It means that two lists are comparable if one can be obtained from the other by removing some elements. In particular, two lists of same length are comparable only if they are equal. By a similar reasoning as in the previous examples, restricting this preorder to all $T\underline{n}$ and computing \sqsubseteq gives the same order.

The previous examples suggest a correspondence between orders \bar{T} on T and preorders on the associated signature as in Def. 7. This is indeed the case.

Proposition 12. *Let T be a finitary Set-functor. Then there is a bijective correspondence between orders on T and presentations with preorders.*

3.4 Lifting T to \hat{T} Using Relators

Starting from an order \bar{T} on T , we will see in this section how a weak pullback preserving T lifts to a Preord-endofunctor \hat{T} using relators. We present below a very brief overview on relators, for more details we refer to [15] or [8].

Let T be a Set-functor. For two sets X, Y and a relation $R \subseteq X \times Y$, the T -relation lifting of R is

$$\text{Rel}_T(R) = \{(u, v) \in TX \times TY \mid \exists w \in TR . T\pi_1(w) = u \wedge T\pi_2(w) = v\}$$

where π_1, π_2 are the projections $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$. The relation lifting satisfies the following properties ([15]):

1. Equality: $=_{TX} = \text{Rel}_T(=_X)$.
2. Inclusion: if $R \subseteq S$ then $\text{Rel}_T(R) \subseteq \text{Rel}_T(S)$.
3. Composition: if $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then $\text{Rel}_T(S \circ R) \subseteq \text{Rel}_T(S) \circ \text{Rel}_T(R)$, with equality if and only if T preserves weak pullbacks.
4. Inverse images (substitution): given functions $f : X \rightarrow X', g : Y \rightarrow Y'$ and relation $R' \subseteq X' \times Y'$, then

$$\text{Rel}_T((f \times g)^{-1}(R')) \subseteq (Tf \times Tg)^{-1}(\text{Rel}_T(R')) \quad (3.4)$$

with equality if T preserves weak pullbacks.

An immediate consequence is the following: if T preserves weak pullbacks and \leq is a preorder on a set X , then $\text{Rel}_T(\leq)$ is a preorder on TX .

Proposition 13. *Let T be a finitary Set-functor which preserves weak pullbacks. Then for each preordered set (X, \leq) , $\text{Rel}_T(\leq)$ coincides with the preorder \leq on TX constructed in Sect. 3.2.⁷*

⁷ Notice that the only thing needed in Prop.13 is that $\text{Rel}_T(\leq)$ is a preorder. This is of course implicit when T preserves weak pullbacks, as mentioned earlier. We do not know if there are examples when $\text{Rel}_T(\leq)$ is a preorder, for any preordered set (X, \leq) , without requesting T to preserve weak pullbacks.

Assume now that there is an order on T , given by $\bar{T}X = (TX, \sqsubseteq)$. For any relation $R \subseteq X \times Y$, the associated relation on $TX \times TY$ given by

$$\text{Rel}_{\bar{T}}^{\sqsubseteq}(R) = \sqsubseteq \circ \text{Rel}_T(R) \circ \sqsubseteq$$

is usually called a T -relator, or lax T -relation lifting ([15], [8]). Equivalently,

$$(u, v) \in \text{Rel}_{\bar{T}}^{\sqsubseteq}(R) \iff \exists w \in T(R) . u \sqsubseteq T\pi_1(w) \text{ and } T\pi_2(w) \sqsubseteq v$$

The T -relator satisfies the following properties ([8], Lemma 4.2):

1. $\sqsubseteq_{TX} = \text{Rel}_{\bar{T}}^{\sqsubseteq}(=_X)$.
2. If $R \subseteq S$ then $\text{Rel}_{\bar{T}}^{\sqsubseteq}(R) \subseteq \text{Rel}_{\bar{T}}^{\sqsubseteq}(S)$.
3. If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then $\text{Rel}_{\bar{T}}^{\sqsubseteq}(S \circ R) \subseteq \text{Rel}_{\bar{T}}^{\sqsubseteq}(S) \circ \text{Rel}_{\bar{T}}^{\sqsubseteq}(R)$.
4. For any functions $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ and any relation $R' \subseteq X' \times Y'$, $\text{Rel}_{\bar{T}}^{\sqsubseteq}((f \times g)^{-1}(R')) \subseteq (Tf \times Tg)^{-1}(\text{Rel}_{\bar{T}}^{\sqsubseteq}(R'))$.

Definition 14. Let T be a Set-endofunctor. We say that an order on T , given by $\bar{T} : \text{Set} \rightarrow \text{Preord}$, $\bar{T}X = (TX, \sqsubseteq)$,

1. is stable (preserves inverse images) if for any two functions $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ and any relation $R' \subseteq X' \times Y'$,

$$\text{Rel}_{\bar{T}}^{\sqsubseteq}((f \times g)^{-1}(R')) = (Tf \times Tg)^{-1}(\text{Rel}_{\bar{T}}^{\sqsubseteq}(R')). \quad (3.5)$$

2. preserves composition of relations if for any $R \subseteq X \times Y$ and $S \subseteq Y \times Z$,

$$\text{Rel}_{\bar{T}}^{\sqsubseteq}(S \circ R) = \text{Rel}_{\bar{T}}^{\sqsubseteq}(S) \circ \text{Rel}_{\bar{T}}^{\sqsubseteq}(R). \quad (3.6)$$

3. preserves composition of preorders if for any preordered set (X, \leq) ,

$$\text{Rel}_{\bar{T}}^{\sqsubseteq}(\leq) \circ \text{Rel}_{\bar{T}}^{\sqsubseteq}(\leq) \subseteq \text{Rel}_{\bar{T}}^{\sqsubseteq}(\leq). \quad (3.7)$$

Proposition 15. For any order on T , we have (3.5) \Rightarrow (3.6) \Rightarrow (3.7).

Now the purpose of all these preparations is the following

Definition-Proposition 16. Let T be a finitary functor having an order \bar{T} which preserves composition of preorders. Then it lifts to a Preord-endofunctor \hat{T} ([8], Lemma 5.5), given by $\hat{T}(X, \leq) = (TX, \text{Rel}_{\bar{T}}^{\sqsubseteq}(\leq))$.

The functor \hat{T} is locally monotone (defining thus a lifting in the sense of Def. 1): assume $f, g : (X, \leq) \rightarrow (Y, \leq)$ are monotone maps such that $f \leq g$ pointwise. Then for any $u \in TX$, we have $u \sqsubseteq u$ and

$$\sqsubseteq = \text{Rel}_{\bar{T}}^{\sqsubseteq}(=_X) \subseteq \text{Rel}_{\bar{T}}^{\sqsubseteq}((f \times g)^{-1}(\leq)) \subseteq (Tf \times Tg)^{-1}(\text{Rel}_{\bar{T}}^{\sqsubseteq}(\leq))$$

therefore $(Tf(u), Tg(u)) \in \text{Rel}_{\bar{T}}^{\sqsubseteq}(\leq)$.

It follows that any finitary functor having an order which preserves composition of preorders has a lifting⁸ to \mathbf{Preord} . We also point out that having an order which preserves composition of relations (preorders) is not equivalent nor implied by the preservation of weak pullbacks by the functor T itself, unless discrete preorder is involved (see Ex. 17.1 and also Ex. 17.5 and Ex. 17.6). Any polynomial functor has an order which preserves composition of relations (see [8], Def. 4.4 and the following paragraph there), but this property is not necessarily preserved by their quotients (see below Example 17.5).

In all examples below, T is a finitary \mathbf{Set} -functor.

- Example 17.* 1. Assume the ordering on the operations of T is discrete. Then \sqsubseteq is equality and $\mathbf{Rel}_{\sqsubseteq}^{\square}(R) = \mathbf{Rel}_T(R)$. Therefore the order on T is stable iff T preserves weak pullbacks. In this case, the lifting of T to \mathbf{Preord} will be $\hat{T}(X, \leq) = (TX, \mathbf{Rel}_T(\leq))$. In view of Prop. 13, we obtain that $\hat{T} = \tilde{T}$.
2. Assume that the order on the operations of T is indiscrete. We have seen that $\sqsubseteq = TX \times TX$, hence $\mathbf{Rel}_{\sqsubseteq}^{\square}(R) = TX \times TY$, for any $R \subseteq X \times Y$, provided $\mathbf{Rel}_T(R)$ is not empty. Actually, what we only need is that $T(\leq)$ to be non empty, which happens for all (finitary) functors T except for the constant one mapping everything to the empty set. Then (4) holds with equality, hence again we get a lifting by $\hat{T}(X, \leq) = (TX, TX \times TX)$.
3. Let T be now the finite power-set functor \mathcal{P}_f , with inclusion as (pre)order on each $\mathcal{P}_f X$. Then for $R \subseteq X \times Y$, $\mathbf{Rel}_{\mathcal{P}_f X}(R)$ can be described as follows (see for example [15], Thm. 2.3.2):

$$(u, v) \in \mathbf{Rel}_{\mathcal{P}_f X}(R) \Leftrightarrow \forall a \in u \exists b \in v . (a, b) \in R \wedge \forall b \in v \exists a \in u . (a, b) \in R$$

An easy computation shows now that the order $X \mapsto (\mathcal{P}_f X, \sqsubseteq)$ preserves composition (is even stable), hence \mathcal{P}_f lifts to a functor $\hat{\mathcal{P}}_f(X, \leq) = (\mathcal{P}_f X, \mathbf{Rel}_{\hat{\mathcal{P}}_f X}^{\sqsubseteq}(\leq))$ on \mathbf{Preord} , with ordering

$$(u, v) \in \mathbf{Rel}_{\hat{\mathcal{P}}_f}^{\sqsubseteq}(\leq) \Leftrightarrow \forall a \in u \exists b \in v . a \leq b$$

4. For $TX = \{\perp\} + X$, the order from Example 11.4 preserves at least composition of preorders, as the relation extension is $\mathbf{Rel}_T(R) = R \cup \{(\perp, \perp)\}$. The resulting functor \hat{T} will then add a bottom element to any preordered set (X, \leq) .
5. Take now the finitary functor $TX = X^3$. The relation lifting associated to this functor is

$$((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbf{Rel}_{(-)^3}(R) \Leftrightarrow ((x_1, y_1), (x_2, y_2), (x_3, y_3)) \in R^3$$

for $R \subseteq X \times Y$. It is well-known that this functor does not preserve weak pullbacks. Recall that we have introduced the zig-zag preorder on TX (Example 11). This order is not stable nor preserves composition: for stability,

⁸ This is no longer an extension: for discrete sets, $\hat{T}(X, =) = (TX, \sqsubseteq)$ is not necessarily discrete.

take $X = \{0\}$, $Y = \{1\}$ and $X' = Y' = \{0, 1\}$, with f, g the inclusion maps, and relation $R' = =_{X'}$. Then $(Tf \times Tg)^{-1}(\text{Rel}_T^{\sqsubseteq}(\equiv)) = TX \times TY$, while $\text{Rel}_T^{\sqsubseteq}((f \times g)^{-1}(\equiv)) = \emptyset$. For preservation of composition, take again $X = \{0, 1\}$ and the (preorder) relation $R = \{(0, 0), (0, 1), (1, 1)\}$. Then for example $((0, 0, 1), (0, 1, 0)) \in \text{Rel}_{(-)_2}^{\sqsubseteq}(R) \circ \text{Rel}_{(-)_2}^{\sqsubseteq}(R)$, but $((0, 0, 1), (0, 1, 0)) \notin \text{Rel}_{(-)_2}^{\sqsubseteq}(R \circ R) = \text{Rel}_{(-)_2}^{\sqsubseteq}(R)$.

6. For the polynomial functor $TX = \mathbb{N} \times X$ lexicographically ordered, a similar argument to the one in [8] shows that it is not stable. But it preserves composition with respect to preorders: if (X, \leq) is a preordered set, then $\text{Rel}_T^{\sqsubseteq}(\leq)$ is again a preorder, namely the usual lexicographic one on $\mathbb{N} \times X$:

$$((n, x), (m, y)) \in \text{Rel}_T^{\sqsubseteq}(\leq) \Leftrightarrow n < m \text{ or } (n = m \text{ and } x \leq y)$$

Proposition 18. *Let T be a finitary Set-functor having an order $\bar{T}(X, \leq) = (TX, \sqsubseteq)$. Then the following are equivalent:*

1. *The order is stable.*
2. *\bar{T} maps weak pullbacks to exact squares.*
3. *The lifted functor \hat{T} preserves exact squares.*

Corollary 19. *Let T be a finitary Set-functor which preserves weak pullbacks. Then the Preord-lifting \tilde{T} from (3.2) preserves exact squares.*

Intuitively, we could simply say that if a Set-functor T preserves exact squares, then its lifting \tilde{T} does so.

It is well known that all Set-functors preserve injective maps with non-empty domain. In Preord we are more interested in embeddings, and they are preserved if the functor preserves exact squares, see [7]. Hence we have the following.

Proposition 20. *If the order on T is stable, then \hat{T} preserves embeddings.*

Corollary 21. *If T preserves weak pullbacks, then \tilde{T} preserves embeddings.*

3.5 Preorder on the Final Coalgebra

There are several papers in the literature describing order relations on the final T -coalgebra (see for example [3] or [8]); as it is expected, there is a connection with the order on the final coalgebra of the lifted functor, first emphasized in [12]. This section intends to present a direct approach of that.

We shall assume that T has a stable order $\bar{T}X = (TX, \sqsubseteq)$ and we shall work with the associated lifting $\hat{T}(X, \leq) = (TX, \text{Rel}_T^{\sqsubseteq}(\leq))$. Recall from [8] that a T -simulation with respect to the order \sqsubseteq between two coalgebras $X \xrightarrow{c} TX$ and $Y \xrightarrow{d} TY$ is a relation $R \subseteq X \times Y$ such that $(x, y) \in R \Rightarrow (c(x), d(y)) \in \text{Rel}_T^{\sqsubseteq}(R)$. In particular, for any \hat{T} -coalgebra $(X, \leq) \xrightarrow{c} (TX, \text{Rel}_T^{\sqsubseteq}(\leq))$, the monotonicity of c implies the preorder \leq is a simulation on X . The greatest simulation between two coalgebras is called similarity and denoted by \lesssim . It satisfies the following ([8], Prop. 5.4):

1. For any T -coalgebra homomorphisms $X \xrightarrow{f} Z$, $Y \xrightarrow{g} W$ and $x \in X, y \in Y$, we have $x \lesssim y \Leftrightarrow f(x) \lesssim g(y)$.
2. Similarity on a coalgebra $X \xrightarrow{c} TX$ is a preorder.

Let now $Z \xrightarrow{\tilde{z}} TZ$ be the final T -coalgebra (which exists by the finitariness assumption on T). As \hat{T} is also finitary and \tilde{U} preserves limits, the final \hat{T} -coalgebra is also Z , but now with some preorder \leq_Z such that $(Z, \leq_Z) \xrightarrow{\tilde{z}} (TZ, \text{Rel}_{\hat{T}}^{\sqsubseteq}(\leq_Z))$ is an isomorphism in \mathbf{Preord} . In particular, \leq_Z is a simulation, hence $\leq_Z \subseteq \lesssim_Z$. Take $(X, \leq) \xrightarrow{c} (TX, \text{Rel}_{\hat{T}}^{\sqsubseteq}(\leq))$ a \hat{T} -coalgebra, with (monotone) anamorphism $(X, \leq) \xrightarrow{!} (Z, \leq_Z)$. By (1) above, $(X, \leq) \xrightarrow{!} (Z, \lesssim_Z)$ is monotone and a \hat{T} -coalgebra map, hence in the following diagram the identity on Z is monotone⁹, implying $\lesssim_Z \subseteq \leq_Z$.

$$\begin{array}{ccc} (X, \leq) & \xrightarrow{!} & (Z, \lesssim_Z) \\ & \searrow & \downarrow \text{id} \\ & & (Z, \leq_Z) \end{array}$$

We have thus the following:

Proposition 22. *The preorder on the final \hat{T} -coalgebra is the similarity.*¹⁰

Remark 23. By [8], Thm. 6.2, if the order satisfies the condition

$$\text{Rel}_{\hat{T}}^{\sqsubseteq}(R_1) \cap \text{Rel}_{\hat{T}}^{\sqsubseteq, \text{op}}(R_2) \subseteq \text{Rel}_T(R_1 \cap R_2) \quad (3.8)$$

for any two relations $R_1, R_2 \subseteq X \times Y$, then the two-way similarity $\lesssim \cap \lesssim^{\text{op}}$ is the same as bisimilarity. This holds for all coalgebras, in particular for the final coalgebra. But bisimilarity on final coalgebra is equality, hence the above condition implies that similarity on the final coalgebra Z is a partial order.

3.6 Third Construction: Order the Variables and Operations

Here we lift an order \bar{T} on T to a \mathbf{Preord} -endofunctor \tilde{T} even in the case that T does not preserve weak pullbacks. The idea is to subsume the constructions in Sect 3.2 and 3.3 in a single one: putting order on the signature (as in Def. 7) and building the coequalizer of (2.3) in \mathbf{Preord} .

Definition 24. *Let T be a functor with preorder \sqsubseteq . Denote by \tilde{T} the functor given by the coequalizer (TX, \preceq) below*

$$\coprod_{m, n < \omega} \text{Set}(\underline{m}, \underline{n}) \times (T\underline{m}, \sqsubseteq) \times (X^n, \leq) \xrightarrow[\lambda]{\rho} \coprod_{n < \omega} (T\underline{n}, \sqsubseteq) \times (X^n, \leq) \xrightarrow{\pi} (TX, \preceq)$$

⁹ As (Z, \lesssim_Z) is a \hat{T} -coalgebra, there is unique monotone map from it to the final \hat{T} -coalgebra. Via the forgetful functor, this is mapped to the unique T -coalgebra map $Z \rightarrow Z$ which is obviously the identity.

¹⁰ See also [13], Thm. 4.1 and [17], Thm. 5.6.

where the domain and the codomain of the coequalizer pair carry the coproduct preorder and each component of the coproduct has the product preorder (where $\text{Set}(\underline{m}, \underline{n})$ is discrete, while X^n has the product preorder obtained from the one on X).

Observe that \tilde{T} is locally monotone and that $\sqsubseteq \subseteq \preceq$. Moreover, we have

Theorem 25. *Let T be a finitary Set-endofunctor which preserves weak pullbacks and has an order which preserves composition of preorders. Then the liftings \tilde{T} and \hat{T} coincide.*

Example 26. The construction presented at the beginning of this section says that we can still get a lifting, independently of T preserving weak pullbacks or the order preserving composition of relations; for example, consider the functor $TX = X_2^3$ with the zigzag order. The corresponding preorder \preceq can then be described as follows: for $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X_2^3$, $(x_1, x_2, x_3) \preceq (y_1, y_2, y_3)$ if $x_i \leq y_i, \forall 1 \leq i \leq 3$, or $x_1 = x_2 = x_3$ and there is some $1 \leq i \leq 3$ with $x_i \leq y_i$.

4 From Preord to Poset

Given a finitary Set-functor T , assume that we have an extension (or a lifting) Γ to Preord. We can move further to Poset by taking the locally monotone functor $T' = Q\Gamma J : \text{Poset} \rightarrow \text{Preord}$, where $Q \dashv J$ is the (monadic) adjunction between the quotient and the inclusion functor mentioned in the preliminaries. If Γ is finitary, then T' is also, since both J and Q preserve filtered colimits.¹¹ Regarding coalgebras, notice that each Γ -coalgebra can be quotiented to a T' -coalgebra, thus there is a functor $Q' : \text{Coalg}(\Gamma) \rightarrow \text{Coalg}(T')$, which sends a Γ -coalgebra $X \xrightarrow{c} \Gamma X$ to $QX \xrightarrow{Qc} Q\Gamma X \rightarrow T'QX$.

Now the discussion bifurcates according to Γ being an extension or a lifting.

If Γ is an extension, a simple computation shows that T' maps discrete sets to discrete sets, thus it is a Poset-extension of T . Moreover, a similar discussion to the one in Section 3.1 shows that the final T' -coalgebra exists and is discrete, with same carrier as the final T -coalgebra, once we assume T (and Γ) finitary.

For the particular extension $\tilde{T}(X, \leq) = (TX, \underline{\leq})$ of a finitary functor T from Section 3.2, relations (3.1) and (3.2), let $T'(X, \leq)$ be the quotient Poset-functor, whose ordering will be denoted for convenience with same symbol $\underline{\leq}$. In case T preserves weak pullbacks, $\underline{\leq}$ on TX has been expressed in terms of the relation lifting (Prop. 13). An immediate result is

Proposition 27. *Let T be a finitary weak pullbacks preserving Set-functor and $T' = Q\tilde{T}J$ its Poset-extension, with \tilde{T} as in (3.1). Then T' preserves exact squares and embeddings.*

Remark 28. If T does not preserve weak pullbacks, then T' may fail to preserve embeddings, as we can see from the following example: take T to be the functor

¹¹ Q is left adjoint, while for J it follows from [6], vol. 2, Prop. 5.5.6.

part of the Boolean algebra monad. On finite sets, we can identify T with the composition of the contravariant power-set functor with itself. Then T does not preserve weak pullbacks [14]. We shall show that the corresponding extension to \mathbf{Poset} does not preserve embeddings. For this, take the embedding of the discrete two-elements poset $\{a, b\}$ into the poset $\{a, b, c\}$ ordered by $a < c, b < c$. Then $T'(\{a, b\}, =)$ is the (discrete) free Boolean algebra on two generators with 16 elements. For the poset (Boolean algebra) $(T'(\{a, b, c\}, \leq), \sqsubseteq)$, notice that monotonicity of operations implies $\perp \sqsubseteq a \wedge \neg a \sqsubseteq a \wedge \neg c \sqsubseteq c \wedge \neg c = \perp$, thus $a \sqsubseteq c$. Similarly, $\perp \sqsubseteq a \wedge \neg a \sqsubseteq c \wedge \neg a \sqsubseteq c \wedge \neg c = \perp$ implies $c \sqsubseteq a$. Thus the images of a and c in $T'(\{a, b, c\}, \leq)$ coincide; similarly for the images of b and c , which makes us conclude that $T'(\{a, b, c\}, \leq)$ has only 4 elements (the free Boolean algebra on only one generator). Hence T' cannot preserve the embedding $(\{a, b\}, =) \hookrightarrow (\{a, b, c\}, \leq)$.

In case Γ is a lifting, there is a partial order on TX for each (X, \leq) and $T'(X, \leq)$ is the quotient of TX with respect to the equivalence relation induced by that preorder. The resulting functor T' is, in general, no longer a lifting of T to \mathbf{Poset} nor an extension.

However, if we consider a particular lifting of T , namely \hat{T} , with respect to an order \sqsubseteq which is already a partial order, then by restricting to posets we obtain that $\mathbf{Rel}_{\hat{T}}^{\sqsubseteq}(\leq)$ is a partial order on TX (for each poset (X, \leq)), once we assume that \sqsubseteq preserve compositions of preorders and satisfies (3.8). In this case, $Q\hat{T}J$ can be identified with $\hat{T}J$ and defines a lifting of T to \mathbf{Poset} . In the general case, however, the best that we can say is that the analogue of Prop. 10 holds, namely that T' will coincide with T on finite sets \underline{n} , with partial order \sqsubseteq .

Example 29. 1. Let \mathcal{D} be the finite subdistribution functor, $\mathcal{D}X = \{d : X \rightarrow [0, 1] \mid \sum_{x \in X} d(x) \leq 1, |\text{supp}(d)| < \infty\}$, with the pointwise order $d \sqsubseteq d' \Leftrightarrow d(x) \leq d'(x), \forall x \in X$. The corresponding $\hat{\mathcal{D}}$ maps posets to posets (see comments after Def. 11 in [12]).

2. Take now \mathcal{P}_f , the finite powerset functor, with the inclusion order. Then this time the \mathbf{Poset} -functor is indeed a quotient, namely the finite convex power set.

Although T' is not a lifting nor an extension, it still behaves well with respect to exact squares (and embeddings):

Proposition 30. *Let T be a finitary \mathbf{Set} -functor having an order $\bar{T}(X, \leq) = (TX, \sqsubseteq)$ which is stable and $T' = Q\hat{T}J$ as above. Then T' preserves exact squares, thus also embeddings.*

5 Conclusion

Considering the rich body of results on \mathbf{Set} -functors, see eg [4], our investigations suggest that analogous results for functors on preorders or on posets would be of interest to coalgebra. For example, characterizations of functors that preserve

exact squares or embeddings would be of interest. This also links the current paper with investigations on coalgebraic logic over preorders or posets, where first steps have been taken on logics given by predicate liftings in [9] and on Moss's ∇ in [5]. Another direction is to follow the connection with coalgebraic (bi)simulations like in [8,12].

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