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NUMERICAL RESULTS FOR THE UNIDIMENSIONAL HEAT PROPAGATION IN A RIGID CONDUCTOR OF INTEGRAL TYPE

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By the Fourier method we find the formal solution of an initial and boundary value problem concerning the unidimensional heat propagation in a rigid conductor of integral type. We show that under properly chosen initial data the formal solution represents the solution in $C^1(R^+; L^2[0, l]) \cap C^0(R^+; H)$ or in $C^1(R^+ \times [0, l])$ of the formulated problem. In the last case we give an estimation of the error arising in the calculation of the solution.

Then we use the integro-interpolation method and construct a three-parameter difference scheme for our problem. Its stability and order of approximation are searched. For various initial data we give numerical results obtained by means of the Fourier method and/or of the difference scheme.

1. INTRODUCTION. THE STATEMENT OF THE PROBLEM

In some papers [3], [5] – [7] it was shown, by various qualitative methods, that the thermoviscoelastic material of integral type allows the propagation of thermomechanical disturbances. Existence and uniqueness properties of the solution to some boundary initial value problems concerning this material were established in [1], [2]. This paper contains a study based on the Fourier method and on the finite differences method in order to obtain *quantitative* results in the case of the unidimensional heat propagation in a rigid heat conductor, i.e. in the case of the following constitutive equations (see [5]):

$$e(x,t) = c_0 T(x,t) + \int_0^t \dot{c}(s) T(x,t-s) \mathrm{d}s, \tag{1.1}$$

$$q(x,t) = -\frac{1}{\tau} \int_0^t \kappa(s) \frac{\partial T}{\partial x}(x,t-s) \mathrm{d}s. \tag{1.2}$$

Here it is understood that the body is held in the reference configuration with uniform temperature θ_0 for all times $t < 0$. $T = \theta - \theta_0$ denotes the temperature

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difference, e–the internal energy, q–the heat flux, τ– the relaxation time and x–the spatial coordinate. The constitutive functions c, κ: $[0,\infty)\equiv R^+\to R$ of time variable s have the properties (see [7])

$$c_\infty \equiv \lim_{s\to\infty} c(s) \geq c_0 \equiv c(0) > 0,\ \kappa_0 \equiv \kappa(0) > 0,$$

but they still remain too general in order to solve an initial and boundary value problem. Therefore we choose them as follows:

$$c(s) = c_\infty - (c_\infty - c_0)\mathrm{e}^{-s/\tau},\ \kappa(s) = \kappa_0 \mathrm{e}^{-s/\tau}, \tag{1.3}$$

where c_∞, c_0, κ_0 are constants with the properties

$$c_0 > 0,\ M \equiv c_\infty / c_0 \geq 1,\ \kappa_0 > 0. \tag{1.4}$$

Devoid of external heat sources a rigid heat conductor must satisfy the energy balance law

$$\rho_0 \frac{\partial e}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{1.5}$$

where ρ_0 is the mass density.

If we insert (1.1), $(1.3)_1$ into (1.5) we obtain

$$\frac{\partial T}{\partial t} + \frac{1}{\rho_0 c_0}\frac{\partial q}{\partial x} = -\frac{M-1}{\tau}\frac{\partial}{\partial t}\int_0^t \mathrm{e}^{-s/\tau} T(x,t-s)\,\mathrm{d}s. \tag{1.6}$$

Further it can easily show that with the choice $(1.3)_2$ the constitutive equation (1.2) turns into the Cattaneo law of heat conduction

$$\frac{\partial q}{\partial t} + \frac{\kappa_0}{\tau}\frac{\partial T}{\partial x} = -\frac{1}{\tau} q \tag{1.7}$$

with vanishing initial condition $q(x, 0) = 0$.

Now we can formulate our mixed problem **P**: Find the temperature difference field T: $[0, l] \times R^+ \to R$ and the heat flux q: $[0, l] \times R^+ \to R$ which satisfy

(*i*) the equations (1.6), (1.7) on $(0, l) \times (0, \infty)$,

(*ii*) the initial conditions

$$T(x,0) = \Theta(x),\ q(x,0) = 0,\ x \in [0,l] \tag{1.8}$$

and

(*iii*) the boundary conditions

$$q(0,t) = q(l,t) = 0,\ t \in R^+. \tag{1.9}$$

Let $L^2\,[0, l]$ be the space of square Lebesgue integrable functions on $[0, l]$ and

$$H = \{u \in L^2[0,l] | u' \in L^2[0,l]\},$$

i.e. the space of absolutely continuous functions on $[0, l]$ having the first derivative in $L^2\,[0, l]$.

If we invoke the result of existence and uniqueness contained in Theorem 4.1. of [1] we can state that

if $\Theta \in H$, *then there exists a unique solution of the problem* **P** *in* $C^1(R^+; L^2\,[0, l]) \cap C^0(R^+; H)$.

Our aim in the next section is to find this solution by the Fourier method and to give sufficient conditions on the function Θ in order to have a *classical solution* i.e. in $C^1\,(R^+ \times [0, l])$. The results are contained in Lemmas 2.1, 2.2. In Lemma 2.3 we give an estimation of the error in computing the classical solution by means of the series obtained by the Fourier method.

Section 3 is concerned with a three-parameter difference scheme for the problem **P**. We show its stability and indicate the order of approximation. In Section 4 we obtain numerical results for various initial data and we present them graphically.

We note that in the case of a rigid heat conductor of Cattaneo type, i.e. $M = 1$, the problem **P** was searched by Sotskii [9]. He used the Fourier method and obtained the solution as formal series. He also proposed a three-parameter difference scheme for which he proved the stability and established the order of approximation. All results from [9] are contained in the present paper, namely for $M = 1$.

2. THE FORMAL SOLUTION. THE CLASSICAL SOLUTION

In this section we use the Fourier method to solve the problem **P**. We look for solutions of (1.6), (1.7), $(1.8)_2$, (1.9) of the form

$$T(x,t) = \overline{T}(x)\widetilde{T}(t), \;\; q(x,t) = \overline{q}(x)\widetilde{q}(t). \tag{2.1}$$

The insertion of (2.1) into (1.6), (1.7) delivers the existence of two constants α, β, such that

$$\overline{q}' = -\rho_0 c_0 \alpha \overline{T}, \; \overline{T}' = -\frac{\beta}{\kappa_0}\overline{q}, \tag{2.2}$$

$$\dot{\widetilde{T}} + \frac{M-1}{\tau}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathrm{e}^{-s/\tau}\widetilde{T}(t-s)\mathrm{d}s = \alpha\widetilde{q}, \;\; \tau\dot{\widetilde{q}} + \widetilde{q} = \beta\widetilde{T}. \tag{2.3}$$

Here the prime denotes the derivative with respect to the local coordinate x, while the superposed dot denotes the derivative with respect to the time t. In view of $(1.8)_2$ and (1.9), $\overline{q}(x)$ and $\widetilde{q}(t)$ have to satisfy

$$\overline{q}(0)=\overline{q}(l)=0,\ \widetilde{q}(0)=0. \tag{2.4}$$

We must investigate four possible cases: (*i*) $\alpha=\beta=0$, (*ii*) $\alpha=0$, $\beta\neq 0$, (*iii*) $\alpha\neq 0$, $\beta=0$, (*iv*) $\alpha, \beta\neq 0$.

In the cases (*i*), (*ii*) we easily obtain a solution $T_0(x,t)$, $q_0(x,t)$ of the form (2.1) as

$$T_0(x,t)=\frac{a_0}{M}\left[1+(M-1)e^{-\frac{M}{\tau}t}\right],\ q_0(x,t)=0, \tag{2.5}$$

with a_0 = constant.

In the case (*iii*) we obtain the trivial solution $T(x,t)=0$, $q(x,t)=0$.

In the case (*iv*), from (2.2), $(2.4)_1$ we get the problem of eigenvalues and eigenfunctions

$$\overline{q}''-\frac{\rho_0 c_0}{\kappa_0}\alpha\beta\,\overline{q}=0,\ \overline{q}(0)=\overline{q}(l)=0,$$

with the nontrivial solutions

$$-\frac{\rho_0 c_0}{\kappa_0}\alpha_n\beta_n=\frac{\pi^2 n^2}{l^2}\equiv\lambda_n,\ \overline{q}_n(x)=\sin\frac{\pi n x}{l},\ n=1,2,\ldots \tag{2.6}$$

By (2.2) we can now calculate $\overline{T}(x)$ as

$$\overline{T}_n(x)=\frac{\beta_n}{\kappa_0\sqrt{\lambda_n}}\cos\sqrt{\lambda_n}x,\ n=1,2,\ldots \tag{2.7}$$

As usually, corresponding to each n = 1, 2, ... we think of solutions $\widetilde{T}_n(t),\widetilde{q}_n(t)$ to the system (2.3), $(2.4)_2$ in form of exponential functions. Because of the integral term in $(2.3)_1$ we look for them of the form

$$\widetilde{T}_n(t)=b_n e^{-\frac{1}{\tau}t}+a_n e^{\gamma_n t},\ \widetilde{q}_n(t)=c_n e^{-\frac{1}{\tau}t}+d_n e^{\gamma_n t}, \tag{2.8}$$

with a_n, b_n, c_n, d_n = constant, $\gamma_n\neq -1/\tau$. Substituting (2.8) into (2.3) we get

$$b_n=0,\ c_n=-\frac{a_n\beta_n\overline{\lambda}}{(\tau\gamma_n+1)\lambda_n},\ d_n=\frac{a_n\beta_n}{\tau\gamma_n+1}, \tag{2.9}$$

$$\gamma_n = \gamma_n^{\pm} \equiv \frac{M}{2\tau}\left(-1 \pm \sqrt{1-\lambda_n/\widetilde{\lambda}}\right) \equiv \frac{M}{2\tau}\left(-1 \pm \sqrt{\Delta_n}\right), \tag{2.10}$$

where $U_T^2 = \kappa_0/\rho_0 c_0 \tau$ is the velocity of thermal pulses (see [5]), $\overline{\lambda} = (M-1)/\tau^2 U_T^2$ and $\widetilde{\lambda} = M^2/4\tau^2 U_T^2$. The inspection of (2.10) shows that both conditions $\gamma_n^+ \neq -1/\tau$ and $\gamma_n^- \neq -1/\tau$ are simultaneously satisfied if and only if $\lambda_n \neq \overline{\lambda}$. Moreover, $\gamma_n^+ \neq \gamma_n^-$ if and only if $\lambda_n \neq \widetilde{\lambda}$. Note that $\overline{\lambda} \le \widetilde{\lambda}$ and $\overline{\lambda} = \widetilde{\lambda}$ if and only if $M = 2$.

Suppose now that $\lambda_n \neq \widetilde{\lambda}$ and $\lambda_n \neq \overline{\lambda}$. Since $\gamma_n^+ \neq \gamma_n^-$, $\gamma_n^{\pm} \neq -1/\tau$, except for some multiplicative constants we have two linearly independent solutions of (2.3) of the form (2.8)–(2.10). Because of the linearity of the system (2.3), their sum is also a solution. But we are interested in solutions which satisfy $(2.4)_2$ and so we obtain

$$\widetilde{T}_n(t) = a_n\left[\left(\tau\gamma_n^+ + 1\right)e^{\gamma_n^+ t} - \left(\tau\gamma_n^- + 1\right)e^{\gamma_n^- t}\right],\ \widetilde{q}_n(t) = a_n\beta_n\left(e^{\gamma_n^+ t} - e^{\gamma_n^- t}\right). \tag{2.11}$$

In the case $\lambda_n \neq \widetilde{\lambda}$, $\lambda_n = \overline{\lambda}$, one of the two γ_n in (2.10) is equal to $-1/\tau$, while the other one is equal to $-(M-1)/\tau \neq -1/\tau$. Thus the solutions of the form (2.8) are given by

$$\widetilde{T}_n(t) = a_n e^{-\frac{M-1}{\tau}t},\ \widetilde{q}_n(t) = \frac{a_n\beta_n}{2-M}\left(e^{-\frac{M-1}{\tau}t} - e^{-\frac{1}{\tau}}\right.$$

Since $\widetilde{q}_n(0) = 0$, these are the solutions of (2.3), $(2.4)_2$.

If $\lambda_n = \widetilde{\lambda}$ and $\lambda_n \neq \overline{\lambda}$, then $\gamma_n^+ = \gamma_n^- = -M/2\tau \neq -1/\tau$ and we seek for solutions of (2.3) of the form

$$\widetilde{T}_n(t) = f_n e^{-\frac{1}{\tau}t} + (a_n + b_n t)e^{-\frac{M}{2\tau}t},\ \widetilde{q}_n(t) = g_n e^{-\frac{1}{\tau}t} + (c_n + d_n t)e^{-\frac{M}{2\tau}t}.$$

We obtain

$$\widetilde{T}_n(t) = a_n\left(\frac{2-M}{2\tau}t + 1\right)e^{-\frac{M}{2\tau}t},\ \widetilde{q}_n(t) = \frac{a_n\beta_n}{\tau}te^{-\frac{M}{2\tau}t}, \tag{2.12}$$

which evidently satisfy $(2.4)_2$.

Finally, if $\lambda_n = \tilde{\lambda} = \bar{\lambda}$ we have $\gamma_n^+ = \gamma_n^- = -1/\tau$ and thus solutions of (2.3) of the form (2.8) do not exist. This time we look for them as

$$\tilde{T}_n(t) = (a_n + b_n t)e^{-\frac{1}{\tau}t}, \quad \tilde{q}_n(t) = (c_n + d_n t)e^{-\frac{1}{\tau}t}$$

and we find

$$\tilde{T}_n(t) = a_n e^{-\frac{1}{\tau}t}, \quad \tilde{q}_n(t) = \frac{a_n \beta_n}{\tau} t e^{-\frac{1}{\tau}t}.$$

These solutions satisfy $(2.4)_2$ and, moreover, they can be simply obtained from (2.12) by setting there $M = 2$.

So if we agree to view the solution (2.5) as corresponding to $n = 0$, we may state that for each $n = 0, 1, \ldots$ we have obtained a solution $T_n(x,t) = \bar{T}_n(x)\tilde{T}_n(t)$, $q_n(x,t) = \bar{q}_n(x)\tilde{q}_n(t)$ of (1.6), (1.7), $(1.8)_2$, (1.9). Supposing $\Theta(x)$ to be in $L^2[0, l]$ with all these solutions let us build up the series $\sum_{n\geq 0} T_n(x,t), \sum_{n\geq 0} q_n(x,t)$ in which the constants $a_0, a_n, \beta_n, n = 1, 2, \ldots$ are choosen so as to have

$$\sum_{n=0}^{\infty} T_n(x,0) = \Theta(x) \tag{2.13}$$

in $L^2[0, l]$. In terms of the cosine Fourier coefficients Θ_n, $n = 0, 1, \ldots$ of $\Theta(x)$, we obtain

$$\sum_{n\geq 0} T_n(x,t) = T_0(t) + \sum_{n\geq 1} T_n(t) \cos\sqrt{\lambda_n}\, x, \tag{2.14}$$

where

$$T_0(t) = \frac{\Theta_0}{2M}\left[1 + (M-1)e^{-\frac{M}{\tau}t}\right], \tag{2.15}$$

$$T_n(t) = \begin{cases} \dfrac{\Theta_n}{2}\left(e^{\gamma_n^+ t} + e^{\gamma_n^- t} + \dfrac{2-M}{M}\,\dfrac{e^{\gamma_n^+ t} - e^{\gamma_n^- t}}{\sqrt{\Delta_n}}\right), & \text{if } \lambda_n \neq \tilde{\lambda}, \lambda_n \neq \bar{\lambda}, \\ \Theta_n e^{-\frac{M-1}{\tau}t}, & \text{if } \lambda_n \neq \tilde{\lambda}, \lambda_n = \bar{\lambda}, \\ \Theta_n\left(\dfrac{2-M}{2\tau}t + 1\right)e^{-\frac{M}{2\tau}t}, & \text{if } \lambda_n = \tilde{\lambda}, n = 1, 2, \ldots \end{cases}$$

and

$$\sum_{n\geq 1} q_n(x,t) = \sum_{n\geq 1} q_n(t)\sin\sqrt{\lambda_n}\,x, \tag{2.16}$$

where

$$q_n(t) = \begin{cases} \dfrac{\kappa_0}{M}\sqrt{\lambda_n}\Theta_n\,\dfrac{e^{\gamma_n^+ t} - e^{\gamma_n^- t}}{\sqrt{\Delta_n}}, & \text{if } \lambda_n \neq \tilde{\lambda}, \lambda_n \neq \bar{\lambda}, \\ \dfrac{\kappa_0}{2-M}\sqrt{\lambda_n}\Theta_n\left(e^{-\frac{M-1}{\tau}t} - e^{-\frac{1}{\tau}t}\right), & \text{if } \lambda_n \neq \tilde{\lambda}, \lambda_n = \bar{\lambda}, \\ \dfrac{\kappa_0}{\tau}\sqrt{\lambda_n}\Theta_n t e^{-\frac{M}{2\tau}t}, & \text{if } \lambda_n = \tilde{\lambda}. \end{cases} \tag{2.17}$$

The series (2.14)–(2.17) are said to be the *formal solution* of the problem **P**.

We learn more about it from the following two lemmas.

Lemma 2.1. *If $\Theta \in H$, then the series (2.14)–(2.17) coverge to the solution in $C^1(R^+; L^2[0,l]) \cap C^0(R^+; H)$ of the problem* **P**.

Proof. To establish this it is sufficient to verify the following properties of the formal solution.

(*i*) The series (2.14)–(2.17) converge uniformly in $L^2[0,l]$ with respect to t on R^+.

The series (2.14), (2.15) has the desired property if, for example, we show that the series $\sum_{n\geq 0} T_n^2(t)$ is uniformly convergent on R^+. An elementary study of $T_n(t)$, $n = 0, 1, \ldots$ delivers the following inequalities on R^+:

$$|T_0(t)| \leq \frac{1}{2}|\Theta_0|, \tag{2.18}$$

$$|T_n(t)| \leq \begin{cases} A|\Theta_n|, & \text{if } \lambda_n \neq \tilde{\lambda}, \lambda_n \neq \bar{\lambda}, \\ |\Theta_n|, & \text{if } \lambda_n \neq \tilde{\lambda}, \lambda_n = \bar{\lambda} \text{ or if } \lambda_n = \tilde{\lambda}, \end{cases} \tag{2.19}$$

where

$$A = 1 + \frac{|M-2|}{M}D, \tag{2.20}$$

$$D=\begin{cases}\dfrac{1}{\sqrt{-\Delta_1}}, & \text{if } \widetilde{\lambda}<\lambda_1<\ldots<\lambda_n<\ldots,\\ \dfrac{1}{\sqrt{-\Delta_2}}, & \text{if } \widetilde{\lambda}=\lambda_1<\ldots<\lambda_n<\ldots,\\ \max\left\{\dfrac{1}{\sqrt{\Delta_{\hat{n}}}},\dfrac{1}{\sqrt{-\Delta_{\hat{n}+1}}}\right\}, & \text{if } \lambda_1<\ldots<\lambda_{\hat{n}}<\widetilde{\lambda}<\lambda_{\hat{n}+1}<\ldots,\\ \max\left\{\dfrac{1}{\sqrt{\Delta_{\tilde{n}-1}}},\dfrac{1}{\sqrt{-\Delta_{\tilde{n}+1}}}\right\}, & \text{if } \lambda_1<\ldots<\lambda_{\tilde{n}-1}<\lambda_{\tilde{n}}=\widetilde{\lambda}<\ldots.\end{cases}$$

From (2.18)–(2.20) it follows that $T_n^2(t)\le A^2\Theta_n^2$, $n=0,1,\ldots$ and, therefor the convergence of the series $\sum_{n\ge0}\Theta_n^2$ implies the uniforme convergence of th series $\sum_{n\ge0}T_n^2(t)$.

Analogously we show that the series (2.16), (2.17) is uniformly convergen in $L^2[0,l]$ with respect to t on R^+ by proving that the series $\sum_{n\ge1}q_n^2(t)$ converge uniformly on R^+. In the same way as for $T_n(t)$, for $q_n(t)$ we obtain the followin inequalities valid on R^+:

$$|q_n(t)|\le\begin{cases}\dfrac{2\kappa_0}{M}D\sqrt{\lambda_n}|\Theta_n|, & \text{if } \lambda_n\ne\widetilde{\lambda},\ \lambda_n\ne\overline{\lambda},\\ \dfrac{2\kappa_0}{|2-M|}\sqrt{\lambda_n}|\Theta_n|, & \text{if } \lambda_n\ne\widetilde{\lambda},\ \lambda_n=\overline{\lambda},\\ \dfrac{2\kappa_0}{Me}\sqrt{\lambda_n}|\Theta_n|, & \text{if } \lambda_n=\widetilde{\lambda}.\end{cases}\tag{2.21}$$

From (2.21) it follows that $q_n^2(t)\le B^2\lambda_n\Theta_n^2$, $n=1,2,\ldots$, where

$$B=\begin{cases}\dfrac{2\kappa_0}{M}\max\left\{D,\dfrac{M}{|2-M|}\right\}, & \text{if } M\ne2,\\ \kappa_0\max\{D,1\}, & \text{if } M=2.\end{cases}\tag{2.22}$$

Since $\Theta\in H$,

$$\sqrt{\lambda_n}\Theta_n=-\Theta'_n,\quad n=1,2,\ldots\tag{2.23}$$

where Θ'_n are the sine Fourier coefficients of Θ' on $[0, l]$. Hence the series $\sum_{n\geq 1}\lambda_n\Theta_n^2$ is convergent and, therefore, the series $\sum_{n\geq 1} q_n^2(t)$ is uniformly convergent.

We conclude that the sums $T(x, t)$, $q(x, t)$ of the series (2.14), (2.15) and (2.16), (2.17), respectively, belong to $C^0(R^+; L^2[0, l])$.

(*ii*) The series obtained by differentiation of (2.14)–(2.17) with respect to t converge uniformly in $L^2[0, l]$ on R^+.

Differentiating the series (2.14)–(2.17) with respect to t, using the inequality

$$|\Delta_n| \leq 1 + \lambda_n / \tilde{\lambda}$$

and the same technical details as in the proof of (*i*) we get the property (*ii*), which implies that $T(x, t)$, $q(x, t)$ are in $C^1(R^+; L^2[0, l])$.

(*iii*) The series obtained by differentiation of (2.14)–(2.17) with respect to x converge uniformly in $L^2[0, l]$ on R^+.

Differentiating the series (2.14)–(2.17) with respect to x and taking into account the inequality

$$\frac{\lambda_n}{|\Delta_n|} < C,\ C = \text{constant} > 1,$$

valid for sufficiently large n, as in (*i*) we obtain the desired property (*iii*). Consequently $T(x, t)$, $q(x, t)$ are in $C^0(R^+; H)$.

(*iv*) The sums of the series (2.14)–(2.17) satisfy the initial conditions (1.8) and the boundary conditions (1.9).

This results obviously from (2.13)–(2.17).

(*v*) The sums of the series (2.14)–(2.17) verify the system (1.6), (1.7).

This is a direct consequence of the properties (*i*), (*ii*), (*iii*) and of the fact that $T_n(x, t)$, $q_n(x, t)$ represent a solution of the system (1.6), (1.7). This completes the proof of Lemma 2.1.

Lemma 2.2 *If* Θ *is an absolutely continuous function on* $[0, l]$, $\Theta' \in H$ *and*

$$\Theta'(0) = \Theta'(l) = 0, \tag{2.24}$$

then the series (2.14)–(2.17) *converge to the classical solution of the problem* **P**.

Proof. First we remark that via (1.7)–(1.9) the relation (2.24) is a necessary condition in order to have a solution in $C^1([0, l] \times R^+)$. Now we verify the following two properties of the formal solution:

(*i*) The series (2.14)–(2.17) converge uniformly on $[0, l] \times R^+$.

As before in (*i*) of Lemma 2.1 we obtain

$$|T_n(x,t)| \leq A|\Theta_n|,\quad |q_n(x,t)| \leq B\sqrt{\lambda_n}|\Theta_n|,\quad n = 1,2,\ldots.$$

By (2.23) we have

$$|\Theta_n| \le \frac{1}{2}\left(|\Theta'_n|^2 + \frac{1}{\lambda_n}\right), \quad n = 1, 2$$

and, hence, the series $\sum_{n\ge0} |\Theta_n|$ is convergent. On the other hand, since $\Theta' \in H$
in view of (2.24) we get

$$\sqrt{\lambda_n}\Theta'_n = \Theta''_n, \quad n = 1,2, \dots \qquad (2$$

where Θ''_n are the cosine Fourier coefficients of Θ'' on $[0, l]$. From (2.23), (2
we deduce that

$$\sqrt{\lambda_n}|\Theta_n| = |\Theta'_n| \le \frac{1}{2}\left(|\Theta''_n|^2 + \frac{1}{\lambda_n}\right).$$

Hence the series $\sum_{n\ge1} \sqrt{\lambda_n}|\Theta_n|$ is also convergent. Consequently the se
(2.14)–(2.17) are uniformly convergent on $[0, l] \times R^+$ and, therefore, their
$T(x, t)$, $q(x, t)$ are in $C^0([0, l] \times R^+)$.

(*ii*) The series obtained by differentiation of (2.14)–(2.17) with respect
and x, respectively, converge uniformly on $[0, l] \times R^+$.

The proof is similar to that of the property (*i*). It follows that $T(x, t)$, $q(x, t)$
in $C^1([0, l] \times R^+)$ and, hence, they represent the classical solution of the probler

It the following we give an estimation of the error in computing the class
solution by means of the series (2.14)–(2.17). We recall that Θ is a piecev
monotone function on $[0, l]$ if there exists the division $0 = \alpha_0 < \alpha_1 < \dots < \alpha_m$
such that Θ is monotone on each (α_i, α_{i+1}). Let $R_n^T(x,t)$ and $R_n^q(x,t)$ the
remainders of the series (2.14), (2.15) and (2.16), (2.17), respectively.

Lemma 2.3. *Under the requirements stated in Lemma 2.2, in which Θ' sh*
be considered as a piecewise monotone function having a finite numbe
discontinuities of the first kind [1], *there hold the inequalities*

$$|R_n^T(x,t)| \le \frac{ACl^2}{2\pi^2 n(n-1)}, \quad |R_n^q(x,t)| \le \frac{BCl}{\pi n}, \quad n = 2,3,\dots, \qquad (2$$

where A, B are given by (2.20), (2.22) *and*

$$C = \frac{4}{\pi}\sum_{i=0}^{m-1}\left(|\Theta''(\alpha_i + 0)| + |\Theta''(\alpha_{i+1} - 0)|\right). \qquad (2$$

[1] Such a function is said to satisfy the Dirichlet conditions.

Proof. From (2.23), (2.25) it follows

$$|\Theta_n| = \frac{1}{\lambda_n}|\Theta_n''|.$$

But the properties of Θ" imply (see [10], 151)

$$|\Theta_n''| \le \frac{C}{n},$$

where C is given by (2.27). Thus we obtain

$$|\Theta_n| \le \frac{Cl^2}{\pi^2 n^3},\ \sqrt{\lambda_n}|\Theta_n| \le \frac{Cl}{\pi n^2}. \tag{2.28}$$

Since

$$|T_n(x,t)| \le A|\Theta_n|,\ |q_n(x,t)| \le B\sqrt{\lambda_n}|\Theta_n|$$

(see the proof of Lemma 2.1), in view of (2.28) we deduce that

$$|R_n^T(x,t)| \le \frac{ACl^2}{\pi^2}\sum_{k=n+1}^{\infty}\frac{1}{k^3},\ |R_n^q(x,t)| \le \frac{BCl}{\pi}\sum_{k=n+1}^{\infty}\frac{1}{k^2}. \tag{2.29}$$

Now in (2.29) we use the inequalities

$$\frac{1}{(k+1)^3} \le \frac{1}{2(k-1)} - \frac{1}{k} + \frac{1}{2(k+1)},\ \frac{1}{(k+1)^2} \le \frac{1}{k} - \frac{1}{k+1}, k = 2,3,\ldots$$

and thus we arrive at the estimations (2.26).

For future use in Section 4 we give the expression of the internal energy e corresponding to the classical solution of the problem **P**:

$$e(x,t) = \frac{c_0}{2}\Theta_0 + \sum_{n=1}^{\infty} e_n(t)\cos\sqrt{\lambda_n}x, \tag{2.30}$$

where

$$e_n(t) = \begin{cases} \frac{c_0}{2}\Theta_n\left[e^{\gamma_n^+ t} + e^{\gamma_n^- t} + \frac{e^{\gamma_n^+ t} - e^{\gamma_n^- t}}{\sqrt{\Delta_n}}\right], & \text{if } \lambda_n \ne \tilde{\lambda}, \lambda_n \ne \bar{\lambda}, \\ \frac{c_0}{M-2}\Theta_n\left[(M-1)e^{-\frac{1}{\tau}t} - e^{-\frac{M-1}{\tau}t}\right], & \text{if } \lambda_n \ne \tilde{\lambda}, \lambda_n = \bar{\lambda}, \\ c_0\Theta_n\left(\frac{M}{2\tau}t + 1\right)e^{-\frac{M}{2\tau}t}, & \text{if } \lambda_n = \tilde{\lambda}. \end{cases} \tag{2.31}$$

3. A THREE-PARAMETER DIFFERENCE SCHEME

The solution of the problem **P** in form of Fourier series was obtained, in fact, for very restrictive initial data, namely for some classes of absolutely continous temperature differences at $t = 0$ (see Lemmas 2.1, 2.2). From this reason it is useful to propose a difference scheme to obtain an approximate solution corresponding to a larger class of initial data. This is precisely what we shall do in the following.

First we remark that for sufficiently smooth solutions the equation (1.6) may be written in the equivalent form

$$\frac{\partial}{\partial t}\left(T + \frac{M-1}{\tau} f\right) + \frac{1}{\rho_0 c_0}\frac{\partial q}{\partial x} = 0, \tag{3.1}$$

$$\frac{\partial f}{\partial t} + \frac{1}{\tau} f - T = 0, \tag{3.2}$$

$$f(x,0) = 0, \ x \in [0,l]. \tag{3.3}$$

In the region $[0, l] \times [0, t_0]$ we choose a net consisting of points $(ih, j\kappa)$, $i = 0, \ldots, N, j = 0, \ldots, J$, and $((i + 1/2)h, j\kappa)$, $i = 0, \ldots, N-1, j = 0, \ldots, J$, where $h = l/N$, $\kappa = t_0/J$. Using the notations

$$y = y_i^j = y(ih, j\kappa), \ \hat{y} = y_i^{j+1}, \ \check{y} = y_i^{j-1}, \ \bar{y} = y_{i+1/2}^j,$$
$$y_x = \frac{1}{h}\left(y_{i+1}^j - y_i^j\right), \ y_{\bar{x}} = \frac{1}{h}\left(y_i^j - y_{i-1}^j\right),$$
$$y_t = \frac{1}{\kappa}(\hat{y} - y), \ y_{\bar{t}} = \frac{1}{\kappa}(y - \check{y}), \ y^{(\sigma)} = \sigma\hat{y} + (1-\sigma)y, \ \sigma \in [0,1],$$

and the integro-interpolation method (see [11], [9]), to the system (3.1), (3.2), (1.7) we associate the following three-parameter difference scheme

$$\bar{T}_t + \frac{M-1}{\tau}\bar{f}_t + \frac{1}{\rho_0 c_0} q_x^{(\sigma_1)} = 0, \tag{3.4}$$

$$q_t + \frac{\kappa_0}{\tau}\bar{T}_{\bar{x}}^{(\sigma_3)} + \frac{1}{\tau} q^{(\sigma_2)} = 0 \tag{3.5}$$

$$\bar{f}_t + \frac{1}{\tau}\bar{f}^{(\sigma_2)} - \bar{T}^{(\sigma_3)} = 0. \tag{3.6}$$

Corresponding to the initial and boundary conditions (1.8), (3.3), (1.9) we choose

$$T^0_{i+1/2} = \frac{1}{h}\int_{ih}^{(i+1)h} \Theta(x)\,\mathrm{d}x,\; f^0_{i+1/2} = 0,\; i = 0,\dots,N-1, \tag{3.7}$$

$$q^0_i = 0,\; i = 0,\dots,N,\; q^j_0 = q^j_N = 0,\; j = 0,\dots,J. \tag{3.8}$$

Here we have supposed $\Theta(x)$ to be integrable on $[0, l]$.

Now we want to compute the solution $\overline{T}$, q of the difference scheme (3.4)–(3.8). First from (3.5), (3.6), $(3.7)_2$, $(3.8)_1$ we deduce that

$$f_{\bar{x}} = -\frac{\tau}{\kappa_0}, \tag{3.9}$$

Then deriving the equation (3.4) with respect to $\bar{x}$, (3.5) with respect to $\bar{t}$ and making use of (3.9), after some calculation we get the following difference equation for q

$$q_{t\bar{t}} - U_T^2\sigma_1\sigma_3\hat{q}_{x\bar{x}} - U_T^2(1-\sigma_1)(1-\sigma_3)\check{q}_{x\bar{x}} -$$

$$U_T^2(\sigma_1+\sigma_3-2\sigma_1\sigma_3)q_{x\bar{x}} + \frac{1}{\tau}q_{\bar{t}}^{(\sigma_2)} + \frac{M-1}{\tau}q_{\bar{t}}^{(\sigma_3)} = 0. \tag{3.10}$$

Using the method of separated variables we find the nonvanishing solution of (3.10), (3.8) as

$$q = \sum_{n=1}^{N-1} b_n^j \sin\frac{\pi n i}{N},\; i = 0,\dots,N,\; j = 0,\dots,J, \tag{3.11}$$

where

$$b_n^j = \begin{cases} \alpha_n\left[\left(q_n^+\right)^j - \left(q_n^-\right)^j\right], & \text{if } \Delta_n > 0,\\ \alpha_n j\left(q_n^+\right)^j, & \text{if } \Delta_n = 0,\\ \alpha_n \mathrm{Im}\left(q_n^+\right)^j, & \text{if } \Delta_n < 0,\; \alpha_n = \text{constant}, \end{cases} \tag{3.12}$$

$$\Delta_n = \frac{\kappa^2}{\tau^2}\left\{\left[M + 4\beta(\sigma_1-\sigma_3)\sin^2\frac{\pi n}{2N}\right]^2 - 16\beta\frac{\tau}{\kappa}\left[1+\frac{\kappa}{\tau}(\sigma_2-\sigma_3)\right]\sin^2\frac{\pi n}{2N}\right\} \tag{3.13}$$

$$q_n^{\pm} = \left[1 - \frac{M\kappa}{2\alpha\tau} - \frac{2\beta\kappa}{\alpha\tau}(\sigma_1+\sigma_3-2\sigma_1\sigma_3)\sin^2\frac{\pi n}{2N} \pm \sqrt{\Delta_n}\right] \Big/ \left(1+\frac{4\beta\kappa}{\alpha\tau}\sigma_1\sigma_3\sin^2\frac{\pi n}{2N}\right), \tag{3.14}$$

$$\alpha = 1+\frac{\kappa}{\tau}\sigma_2+(M-1)\frac{\kappa}{\tau}\sigma_3,\ \beta=\frac{\kappa\kappa_0}{h^2\rho_0 c_0} \tag{3.15}$$

Now the difference equations (3.4), (3.6) give

$$\overline{T}^{j+1} = a\overline{T}^j + \kappa F^{j-1},\ j=1,\ldots,J, \tag{3.16}$$

where

$$F^{j-1} = \frac{1}{\alpha\tau}T^0 + \frac{1}{\alpha}\left(1+\frac{\kappa}{\tau}\sigma_2\right)G^j - \frac{1}{\alpha}\left(1+\frac{\kappa}{\tau}\sigma_2-\frac{\kappa}{\tau}\right)G^{j-1},$$
$$G^{j-1} = -\frac{1}{\rho_0 c_0}\left(q_x^0+\ldots+q_x^{j-1}+\sigma_1 q_x^j\right),\ \ a=1-\frac{M\kappa}{\alpha\tau}, \tag{3.17}$$

while $\overline{T}^1$ is computed by means of (3.4), (3.6) at $j = 0$ and of (3.7) as

$$\overline{T}^1 = \left[a+\frac{\kappa}{\alpha\tau}\right]\overline{T}^0 - \frac{\sigma_1\kappa}{\rho_0 c_0}\left[1-(M-1)\frac{\sigma_3\kappa}{\alpha\tau}\right]q_x^1. \tag{3.18}$$

The solution of (3.16), (3.17) is

$$\overline{T}^j = \begin{cases} a^{j-1}\overline{T}^1 + \sum_{p=0}^{j-2}\kappa a^{j-2-p}F^p, & \text{if } a\neq 0, \\ \kappa F^{j-2}, & \text{if } a=0,\ j=2,\ldots,J. \end{cases} \tag{3.19}$$

In (3.16)–(3.19) we have omitted to write the indices i, $i + 1/2$.

In order to find the constants α_n, $n = 1, \ldots, N-1$ we write the equation (3.5) for $j = 0$ and replace q^1, $\overline{T}^1$ with the values given by (3.11), (3.18). Thus we get the system

$$\sum_{n=1}^{N-1} b_n^1\left(1+\frac{4\beta\kappa}{\alpha\tau}\sigma_1\sigma_3\sin^2\frac{\pi n}{2N}\right)\sin\frac{\pi n i}{N} = -\frac{\kappa_0\kappa}{\alpha\tau h}\left(\overline{T}_{i+1/2}^0-\overline{T}_{i-1/2}^0\right), \tag{3.20}$$

$i = 1, \ldots, N-1$, having a unique solution b_n^1, $n = 1, \ldots, N-1$.

Since

$$e = c_0\left(T+\frac{M-1}{\tau}f\right)$$

and, hence,

$$\bar{e}^j = c_0\left(\bar{T}^j + \frac{M-1}{\tau}\bar{f}^j\right), \tag{3.21}$$

if we want to know the internal energy corresponding to the solution of the problem **P** we must determine $\bar{f}^j$. It is not difficult to show that

$$\bar{f}^1 = \frac{\kappa}{\alpha}\bar{T}^0 - \frac{\kappa\sigma_1\sigma_3}{\rho_0 c_0 \alpha} q_x^1, \tag{3.22}$$

$$\bar{f}^j = \begin{cases} a^{j-1}\bar{f}^1 + \sum_{p=0}^{j-2} \kappa a^{j-2-p} H^p, & \text{if } a \neq 0, \\ \kappa H^{j-2}, & \text{if } a = 0,\ j = 2,\dots,J \end{cases}$$

where

$$H^{j-1} = \frac{1}{\alpha}\left(\bar{T}^0 + \sigma_3 G^j + (1-\sigma_3)G^{j-1}\right), j = 1,\dots,J. \tag{3.23}$$

Now we establish sufficient conditions for the stability of the difference scheme (3.4)–(3.8). The equation (3.10) for q can be put into the form

$$Bq_{\overset{\circ}{t}} + \kappa^2 R q_{\bar{t}t} + Aq = 0, \tag{3.24}$$

where

$$B = \frac{M}{\tau U_T^2}E + \kappa(\sigma_1 + \sigma_3 - 1)\Lambda, \quad A = \Lambda, \quad \Lambda q = -q_{\bar{x}x},$$

$$R = \frac{\alpha}{\kappa^2 U_T^2}\left(1 - \frac{M\kappa}{2\alpha\tau}\right)E + \frac{1}{2}(1 - \sigma_1 - \sigma_3 + 2\sigma_1\sigma_3)\Lambda, \quad q_{\overset{\circ}{t}} = \frac{1}{2}(q_t + q_{\bar{t}}),$$

and E is the identity operator. If $B \geq 0$ and $R - \frac{1}{4}A > 0$, then the difference equation (3.21) is stable with respect to the initial data q^0, q^1 (see [11], **6.3**). Using the inequality (see [11], **2.3.3**)

$$\frac{8}{l^2}\|y\|^2 \leq (\Lambda y, y) \leq \frac{4}{h^2}\|y\|^2,$$

we obtain that if

$$M + 4\beta(\sigma_1 + \sigma_3 - 1) \geq 0, \tag{3.25}$$

then $B \geq 0$. In exactly the same way it results that the following inequalities

$$2\alpha\frac{\tau}{\kappa}-M+16\beta\frac{h^2}{l^2}\left(\sigma_1-\frac{1}{2}\right)\left(\sigma_3-\frac{1}{2}\right)>0,\quad \text{if}\left(\sigma_1-\frac{1}{2}\right)\left(\sigma_3-\frac{1}{2}\right)\geq 0,$$
$$2\alpha\frac{\tau}{\kappa}-M+8\beta\left(\sigma_1-\frac{1}{2}\right)\left(\sigma_3-\frac{1}{2}\right)>0,\quad \text{if}\left(\sigma_1-\frac{1}{2}\right)\left(\sigma_3-\frac{1}{2}\right)<0, \tag{3.26}$$

are sufficient in order to have $R-\frac{1}{4}A>0$.

The difference scheme (3.16)–(3.18) is stable (with respect to the initial data $\overline{T}^0$ and to the right-hand side, in the sense precised in [11], **6.1.5**), if

$$|a|\leq 1,\ \text{viz. } M\frac{\kappa}{\tau}\leq 2\alpha \tag{3.27}$$

(see [11], **6.1.6**, Satz 2 with $L=1/\tau$).

We shall analyse the conditions (3.25)–(3.27) for some particular values of the parameters $\sigma_1, \sigma_2, \sigma_3$.

(*i*) $\sigma_2=\sigma_3=0$. In this case the inequalities (3.25)–(3.27) become

$$M+4\beta(\sigma_1-1)\geq 0,$$
$$\frac{\tau}{\kappa}>\frac{M}{2}+4\beta\frac{h^2}{l^2}\left(\sigma_1-\frac{1}{2}\right)>0,\quad \text{if } \sigma_1\leq\frac{1}{2},$$
$$\frac{\tau}{\kappa}>\frac{M}{2}+2\beta\left(\sigma_1-\frac{1}{2}\right)<0,\quad \text{if } \sigma_1>\frac{1}{2}, \tag{3.28}$$
$$\frac{\tau}{\kappa}\geq\frac{M}{2}.$$

But we want our scheme to be valid for sufficiently small τ ($\tau\approx 10^{-11}$ *sec* for metals, see [8]) and, therefore, the conditions (3.28) can be hardly satisfied. It remains to show otherwise if our scheme is stable or not. Namely, we observe that

$$q_n^+=1-\frac{4\beta\sin^2\frac{\pi n}{2N}}{M+4\beta\sigma_1\sin^2\frac{\pi n}{2N}}+O\left(\frac{\tau}{\kappa}\right),$$
$$q_n^-=-\frac{\kappa}{\tau}\left(M+4\beta\sigma_1\sin^2\frac{\pi n}{2N}\right)+O(1),\ \tau\to 0,$$

and, therefore, $|b_n^j|\to\infty$ if $\tau\to 0$. Consequently the difference equation (3.10) is unstable.

(*ii*) $\sigma_1 = \sigma_2 = \sigma_3 = 1$. It is easy to see that the conditions (3.25)–(3.27) are satisfied for every h, $\kappa > 0$.Thus the difference scheme (3.4)–(3.8) is absolutely stable.

(*iii*) $\sigma_1 = \sigma_2 = \sigma_3 = 1/2$. Again the conditions (3.25)–(3.27) are satisfied for every h, $\kappa > 0$, hence the difference scheme (3.4)–(3.8) is absolutely stable.

Finally, an elementary analysis shows that on sufficiently smooth solutions of the problem **P** the order of approximation of the difference scheme (3.4)–(3.8) is $O(h^2 + \kappa^2)$ if $\sigma_1 = \sigma_2 = \sigma_3 = 1/2$ and $O(h^2 + \kappa)$ otherwise.

4. NUMERICAL RESULTS

It this Section we carry out a numerical examination of the problem treated previously by choosing a rigid conductor whose material constants are $\rho_0 = 1$, $c_0 = 1$, $\kappa_0 = 1$, $\tau = 1$, $M = 1$ and $M = 2$. We suppose $l = 2\pi$.

First we consider the initial datum

$$\Theta(x) = \frac{\theta_0}{l^3}\left(x^3 - \frac{3}{2}lx^2 + l^3\right),\ \theta_0 = 100\ ,$$

which satisfies the requirements of the Lemma 2.2. Therefore, the problem **P** has a classical solution which can be calculated by means of the series (2.14)–(2.17). Since the cosine Fourier coefficients of $\Theta(x)$ have the property

$$|\Theta_n| \le \frac{24\theta_0}{\pi^4 n^4},\ n = 1,2,\ldots,$$

for the remainders $R_n^T(x,t)$, $R_n^q(x,t)$ we can find a better estimation than that delivered by Lemma 2.3, namely

$$\left|R_n^T(x,t)\right| \le \frac{8A\theta_0}{\pi^4 n(n-1)(n-2)},\quad \left|R_n^q(x,t)\right| \le \frac{6B\theta_0}{\pi^4 n(n-1)}.$$

Consequently we obtain that

for $M=1$,	for $M=2$,
$\left\|R_n^T(x,t)\right\| < 10^{-2}$, if $n=11$,	$\left\|R_n^T(x,t)\right\| < 10^{-2}$ if $n=11$,
$\left\|R_n^q(x,t)\right\| < 10^{-2}$, if $n=35$,	$\left\|R_n^q(x,t)\right\| < 10^{-2}$ if $n=27$.

For $n = 11$ the calculated temperature $T((i + 1/2)\,h, jk)$, $i = 0, \ldots, 9$ and heat flux $q(ih, j\kappa)$, $i = 0, \ldots, 10$, $h = \pi/5$, $\kappa = 0.1$, for various j are plotted for $M = 1$ in Figures 1, 2 and for $M = 2$ in Figures 3, 4.

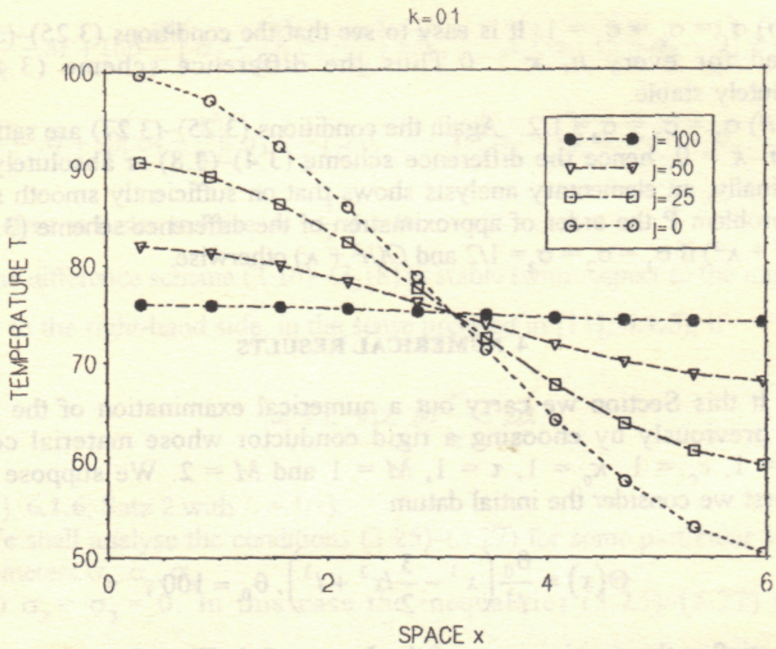

Fig. 1. Variation of temperature (internal energy) with space for $M = 1$.

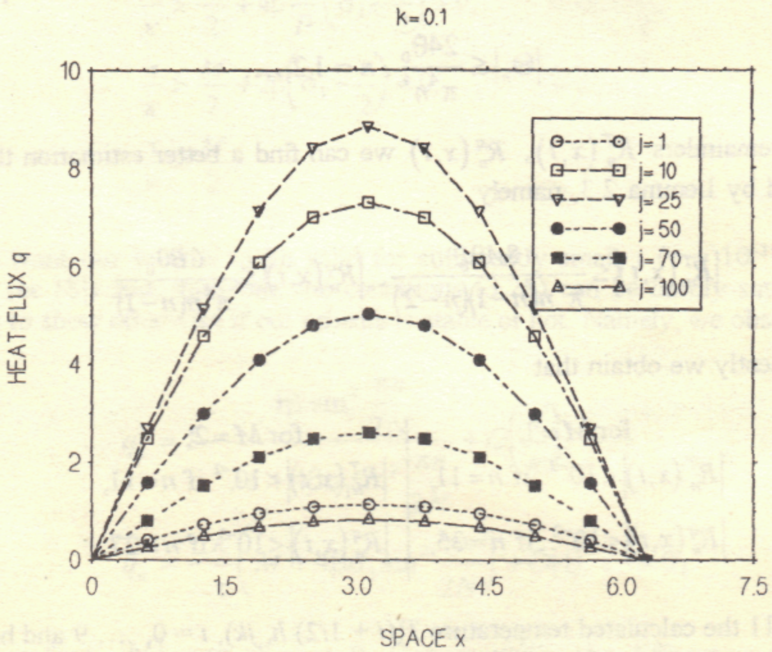

Fig. 2. Variation of heat flux with space for $M = 1$.

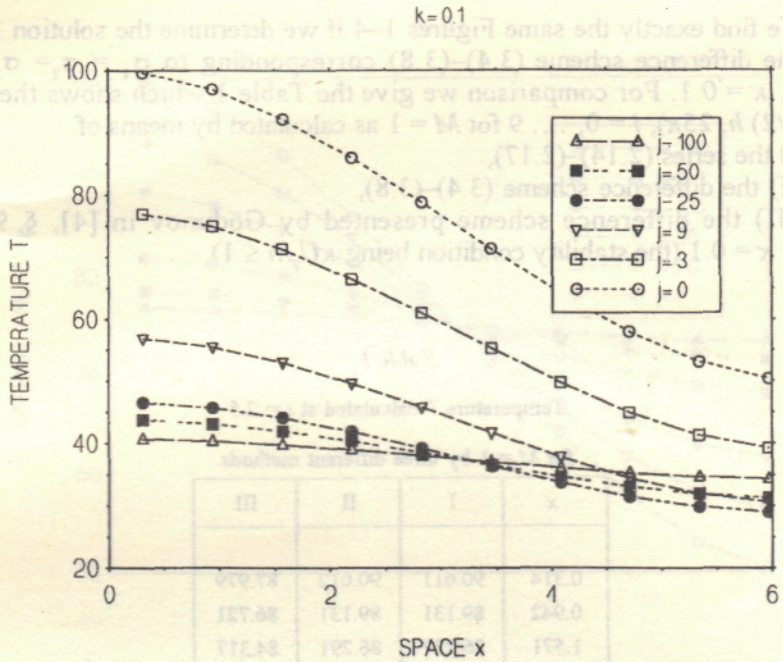

Fig. 3. Variation of temperature with space for $M = 2$.

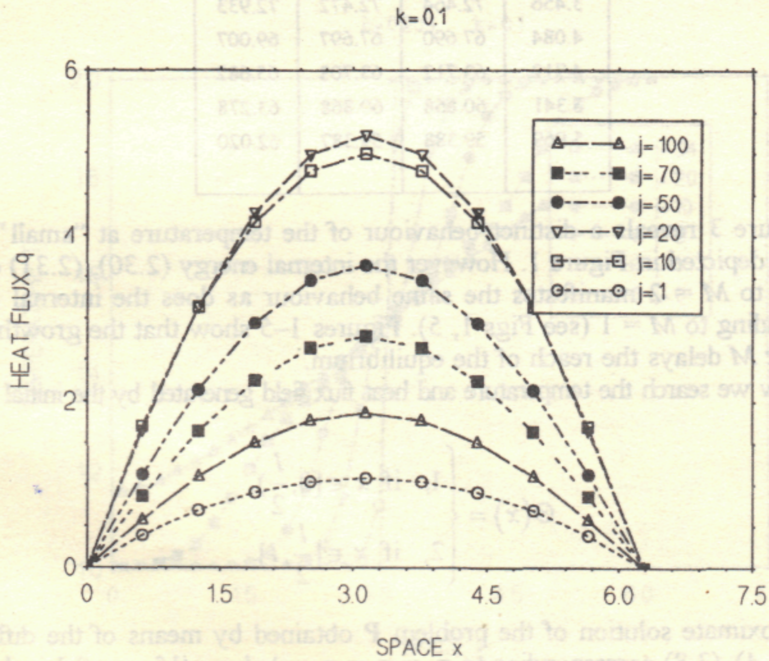

Fig. 4. Variation of heat flux with space for $M = 2$.

We find exactly the same Figures 1–4 if we determine the solution T, q by using the difference scheme (3.4)–(3.8) corresponding to $\sigma_1 = \sigma_2 = \sigma_3 = 0.5$, $h = \pi/5$, $\kappa = 0.1$. For comparison we give the Table 1 which shows the values $T((i + 1/2)\, h, 25\kappa)$, $i = 0, \ldots, 9$ for $M = 1$ as calculated by means of

(I) the series (2.14)–(2.17),

(II) the difference scheme (3.4)–(3.8),

(III) the difference scheme presented by Godunov in [4], § 9, with $h = \pi/5$, $\kappa = 0.1$ (the stability condition being $\kappa U_T/h \leq 1$).

Table 1

Temperature T calculated at $t = 2.5$ for $M = 1$ by three different methods.

x	I	II	III
0.314	90.611	90.612	87.979
0.942	89.131	89.131	86.721
1.571	86.287	86.291	84.317
2.199	82.309	82.302	80.992
2.827	77.535	77.527	77.066
3.456	72.464	72.472	72.933
4.084	67.690	67.697	69.007
4.712	63.712	63.708	65.682
5.341	60.868	60.868	63.278
5.969	59.388	59.387	62.020

Figure 3 reveals a distinct behaviour of the temperature at "small" times from that depicted in Figure 1. However the internal energy (2.30), (2.31) corresponding to $M = 2$ manifestes the same behaviour as does the internal energy corresponding to $M = 1$ (see Figs 1, 5). Figures 1–5 show that the growth of the parameter M delays the reach of the equilibrium.

Now we search the temperature and heat flux field generated by the initial datum

$$\Theta(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{l}{2}), \\ 2, & \text{if } x \in [\frac{l}{2}, l]. \end{cases}$$

The approximate solution of the problem **P** obtained by means of the difference scheme (3.4)–(3.8) corresponding to $\sigma_1 = \sigma_2 = \sigma_3 = 1$, $h = \pi/15$, $\kappa = 0.1$ is depicted for $M = 1$ in Figures 6, 7 and for $M = 2$ in Figures 8, 9.

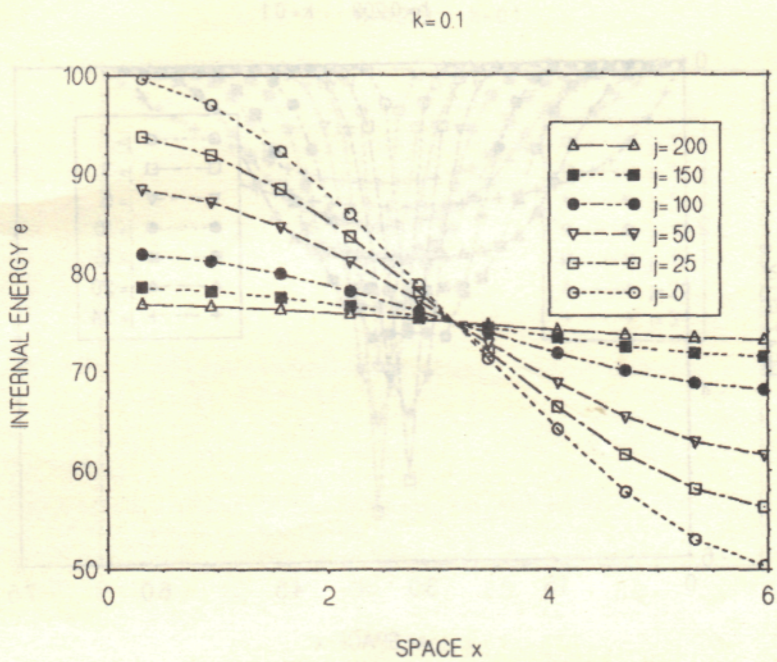

Fig. 5. Variation of internal energy with space for $M = 2$.

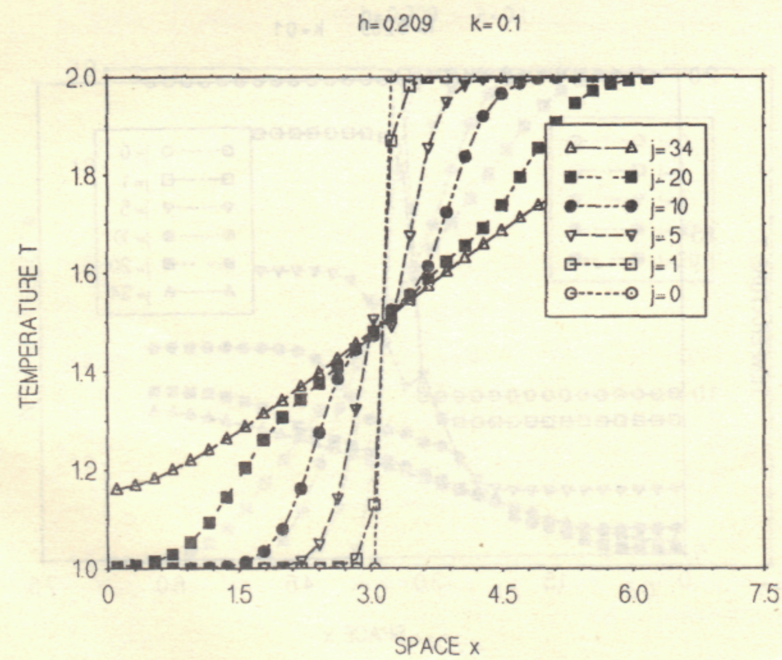

Fig. 6. Variation of temperature (internal energy) with space for $M = 1$.

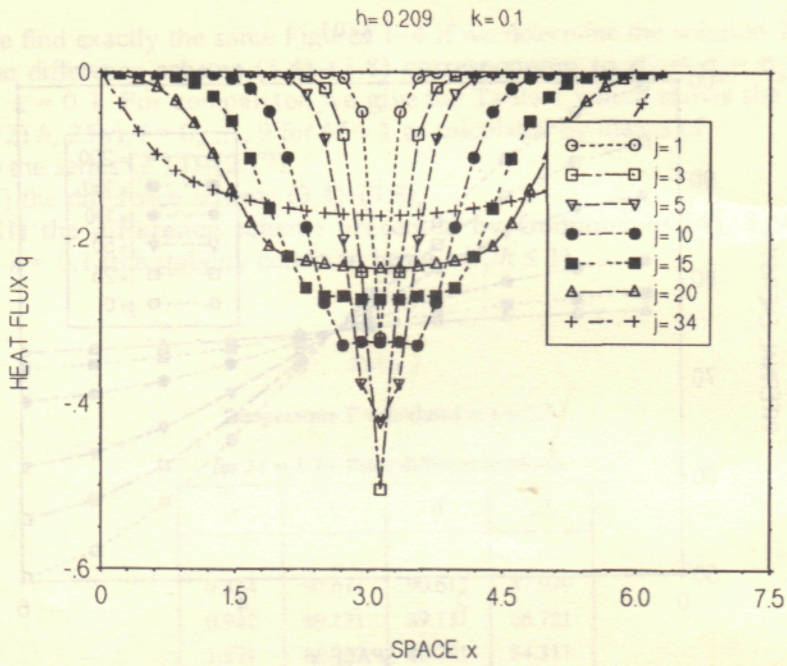

Fig. 7. Variation of heat flux with space for $M = 1$.

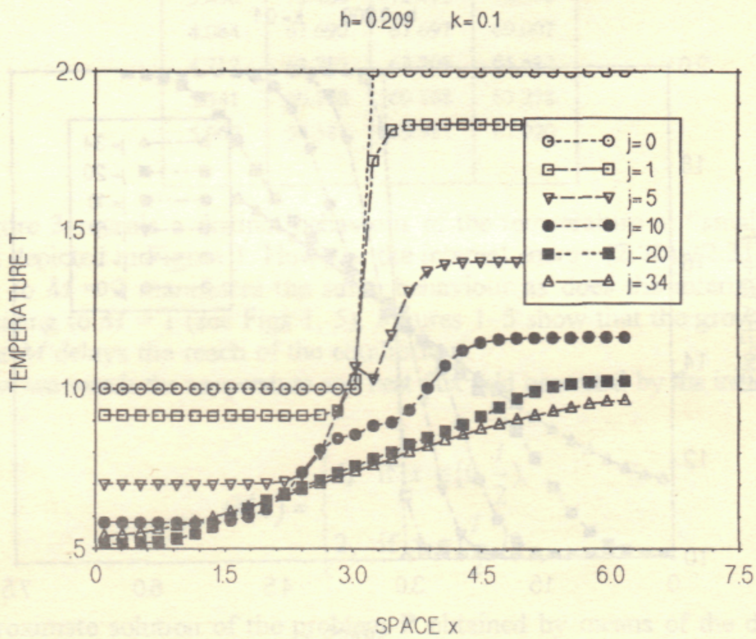

Fig. 8. Variation of temperature with space for $M = 2$.

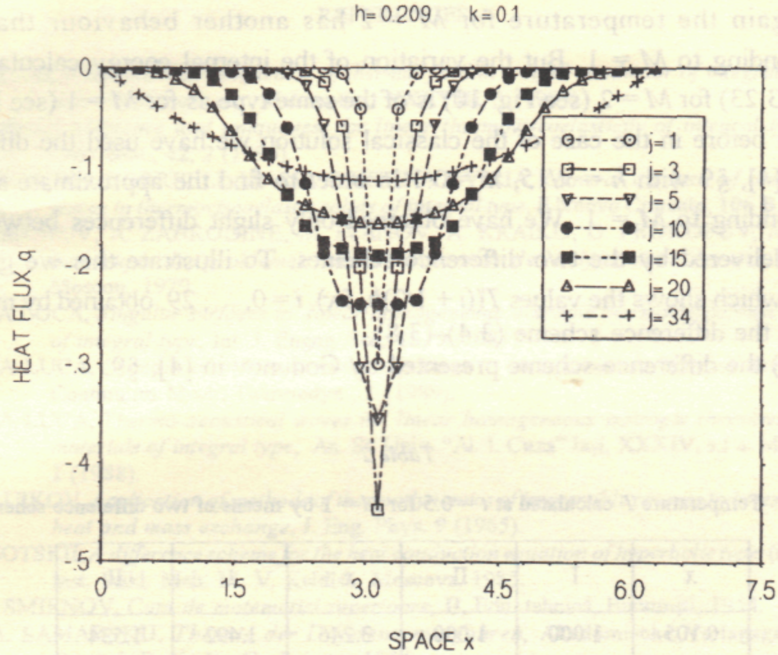

Fig. 9. Variation of heat flux with space for $M = 2$.

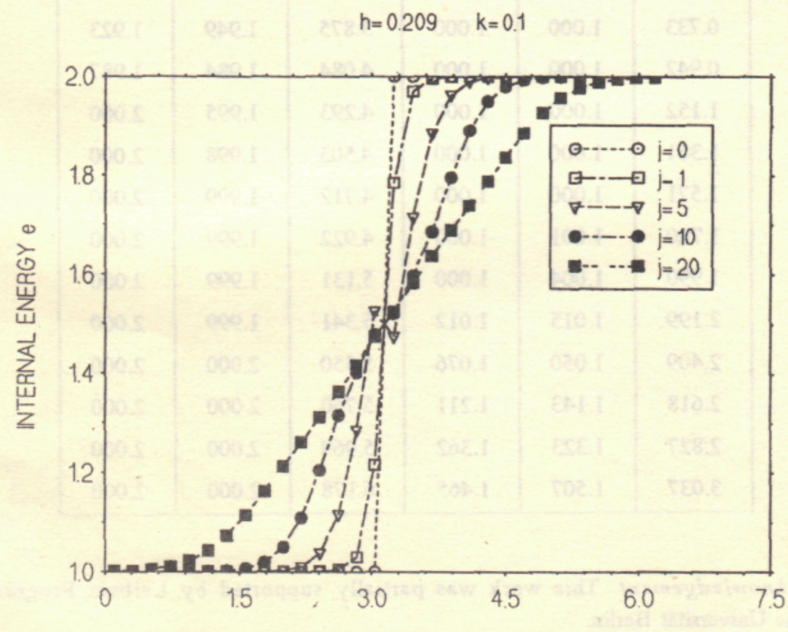

Fig. 10. Variation of internal energy with space for $M = 2$.

Again the temperature for $M = 2$ has another behaviour than that corresponding to $M = 1$. But the variation of the internal energy calculated via (3.21)–(3.23) for $M = 2$ (see Fig. 10) is of the same type as for $M = 1$ (see Fig. 6).

As before in the case of the classical solution we have used the difference scheme [4], §9 with $h = \pi/15$, $\kappa = 0.1$ in order to find the approximate solution corresponding to $M = 1$. We have obtained only slight differences between the results delivered by the two difference schemes. To illustrate this we give the Table 2 which shows the values $T((i + 1/2)h, 5\kappa)$, $i = 0, \ldots, 29$, obtained by means of

(I) the difference scheme (3.4)–(3.8),

(II) the difference scheme presented by Godunov in [4], §9.

Table 2

Temperature T calculated at $t = 0.5$ for $M = 1$ by means of two difference schemes

x	I	II	x	I	II
0.105	1.000	1.000	3.246	1.492	1.534
0.314	1.000	1.000	3.456	1.676	1.637
0.524	1.000	1.000	3.665	1.856	1.788
0.733	1.000	1.000	3.875	1.949	1.923
0.942	1.000	1.000	4.084	1.984	1.987
1.152	1.000	1.000	4.293	1.995	2.000
1.361	1.000	1.000	4.503	1.998	2.000
1.571	1.000	1.000	4.712	1.999	2.000
1.780	1.001	1.000	4.922	1.999	2.000
1.990	1.004	1.000	5.131	1.999	2.000
2.199	1.015	1.012	5.341	1.999	2.000
2.409	1.050	1.076	5.550	2.000	2.000
2.618	1.143	1.211	5.760	2.000	2.000
2.827	1.323	1.362	5.969	2.000	2.000
3.037	1.507	1.465	6.178	2.000	2.000

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